Descriptive set theory and classification problems in C*-algebra theory

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1. **Descriptive set theory and classification**
   - **Classification problems**
   - **Hierarchy**
2. **Non-classification tools**
   - **Reduction**
   - **Proper generic ergodicity**
   - **Generic turbulence**

Classification problems

Suppose that:

- $\mathcal{C}$ is a class of **objects**;
- $\mathcal{R}$ is an **equivalence relation** on $\mathcal{C}$.

Classifying the objects of $\mathcal{C}$ up to $\mathcal{R}$ usually means assigning (in a "constructive" way) to every element of $\mathcal{C}$ some "simple" invariant, in such a way that two objects are related by $\mathcal{R}$ if and only if the associated invariants are equal (or equivalent in some sense).

E.g., the class of **AF algebras** is classified up to $*$**-isomorphism** by the K_0 group, since

- $K_0(A)$ is constructively associated to A;
- two AF algebras A and B are isomorphic iff $K_0(A)$ and $K_0(B)$ are isomorphic as dimension groups (Elliott, 1976).

Descriptive set-theoretic framework

Descriptive set theory offers a framework where one can study and compare the **complexity of different classification problems**.

Suppose, as before, that

- $\mathcal{C}$ is a class of **objects**;
- $\mathcal{R}$ is an **equivalence relation** on $\mathcal{C}$.

One wants to study how difficult it is to classify the elements of $\mathcal{C}$ up to $\mathcal{R}$.

Objects in $\mathcal{C}$ are **parametrized** by points in some **Polish space** X.

The relation $\mathcal{R}$ on $\mathcal{C}$ induces **a relation E on X**.

The parametrization is "good" if E is **Borel** (or at least analytic).

Examples of relations and parametrizations

For example, suppose that

- $\mathcal{C}$ is the class of **separable C*-algebras**;
- $\mathcal{R}$ is the relation of $*$**-isomorphism**.

In this case, if H is the separable Hilbert space, the elements of $\mathcal{C}$ can be parametrized by sequences of elements in the unit ball of $B(H)$, where

- the unit ball of $B(H)$ is endowed with the **weak operator topology**;
- the set of sequences of elements in the unit ball of $B(H)$ is a Polish space with respect to the **product topology**.

As another example, suppose that

- $\mathcal{C}$ is the set $\mathrm{Aut}(A)$ of **automorphisms of a C*-algebra** A;
- $\mathcal{R}$ is the relation of **unitary equivalence**.

$\mathrm{Aut}(A)$ is a Polish space with respect to the **point-norm topology**.

Borel reduction

One now wants to study and compare pairs (X, E), where

- X is a Polish space;
- E is an analytic equivalence relation.

A **notion of comparison** is given by (Borel) reduction.

Definition

A (Borel) reduction from (X, E) to (X', E') is a function $f : X \to X'$ s.t.

- f is Borel;
- for $x, y \in X$, xEy iff $f(x) E' f(y)$.

If there is a (Borel) reduction from (X, E) to (X', E'),
we say that (X, E) is **(Borel) reducible** to (X', E'), and we write

$$(X, E) \leq_B (X', E').$$

Benchmark equivalence relations

These are benchmark relations to be used as **measures of complexity:**

1. the relation $=_{\mathbb{R}}$ of **equality of real numers**;
2. the **isomorphism** relation $E^{\mathcal{C}}_{\simeq}$ in a class $\mathcal{C}$ of **countable structures** (such as groups, rings etc...);
3. the **orbit equivalence relation** E^G_X induced by a continuous group action of a Polish group G on a Polish space X;
4. the relation $E^{Banach}_{\simeq}$ of **topological isomorphism of separable Banach spaces**.

These (families of) relations give a hierarchy of complexity, since

$$=_{\mathbb{R}} \leq_B E^{\mathcal{C}}_{\simeq} \leq_B E^{S_\infty}_{\mathbb{N}}$$

Moreover, any orbit equivalence relation and, in fact,
any analytic equivalence relation is reducible to $E^{Banach}_{\simeq}$.

Hierarchy of complexity

Definition

An equivalence relation E is called

1. **smooth** if it is reducible to $=_{\mathbb{R}}$;
2. **classifiable by countable structures** if it is reducible to $E_{\simeq}^{\mathcal{C}}$ for some class $\mathcal{C}$ of countable structures;
3. **below a group action** if it is reducible to E_X^G for some continuous action of a Polish group G on a Polish space X;
4. **complete analytic** if $E_{\simeq}^{Banach}$ (or, equivalently, any analytic equivalence relation) is reducible to it.

Examples of relations of different complexity

Isomorphism and bi-embeddability of various classes of **separable C*-algebras** offer examples of relations of different degrees of complexity:

1. isomorphism of **UHF algebras** is smooth (Glimm, 1960: they are classified by the associated supernatural number);
2. **Kirchberg algebras in the UCT class** are classifiable up to isomophism by countable structures (Kirchberg-Phillips, 1994: they are classified by the K_0 and K_1 groups) but not smooth;
3. isomorphism of **simple nuclear unital C*-algebras** is below a group action of $\mathrm{Aut}(\mathcal{O}_2)$, but it is not classifiable by countable structures (Farah-Toms-Törnquist, 2011);
4. **bi-embeddability** of **AF algebras** is not below a group action (Farah-Toms-Törnquist, 2011).

Open problems

The exact picture is still unclear, and there are many open problems:

- is bi-embeddability of AF algebras complete analytic?
- what about bi-embeddability of arbitrary C*-algebras?
- is isomorphism of (simple, unital) C*-algebras below a group action?

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   - **Generic turbulence**

Non-classification tools

Tools to prove that some relation E is not classifiable (in some sense):

- **reduction** from a relation already known to be non-classifiable to E;
- **proper generic ergodicity** (to prove non-smoothness of an orbit equivalence relation);
- **generic turbulence** (to prove non classifiability by countable structures of an orbit equivalence relation).

E_0

Let E_0 be the relation of **eventual equality** of sequences of **0's and 1's**.

E_0 is a useful test relation for non-smoothness, in view of the following

Theorem (Harrington-Kechris-Loveau, 1990)

If E_0 can be reduced to some equivalence relation E, then E is not smooth. The converse holds if E is Borel.

For example, E_0 is reducible to isomorphism of AF algebras.
Hence, the latter relation is not smooth.

E_1

Denote by E_1 the relation of **eventual equality** of sequences in $\mathbb{R}$.

A relation to which E_1 is reducible is not below a group action.

Proposition (Farah-Toms-Tørnquist, 2011)

E_1 is reducible to the relation of bi-embeddability of AF algebras. In particular, the latter relation is not below a group action.

Fix an enumeration $(p_{n,q})_{(n,q)\in\mathbb{N}\times\mathbb{Q}}$ of the primes. The reduction is

$$\begin{aligned} \mathbb{R}^{\mathbb{N}} &\to \{\text{AF algebras}\} \\ (x_n)_{n\in\mathbb{N}} &\mapsto \bigoplus_{k\in\mathbf{N}} \left(\bigotimes_{n\leq k} \bigotimes_{q\in\mathbb{Q}} \mathbb{M}_{p_{n,q}^{\infty}} \otimes \bigotimes_{n>k} \bigotimes_{q<x_n} \mathbb{M}_{p_{n,q}} \right) \end{aligned}$$

Strongly dense groups

Suppose that G is a **Polishable subgroup** of $\mathbb{T}^{\mathbb{N}}$.

We say that G is super dense if, for every $n \in \mathbb{N}$ and $(y_1, \ldots, y_n) \in \mathbb{T}^n$ there is $(z_i)_{i\in\mathbb{N}} \in G$ such that $z_i = y_i$ for $1 \leq i \leq n$.

For example, ℓ_p for $p \in [1, +\infty)$ and c_0 are super dense, where

$$\ell_p = \left\{ (z_i)_{i\in\mathbb{N}} \in \mathbb{T}^{\mathbb{N}} \,\middle|\, \sum_{i\in\mathbb{N}} d(z_i, 1)^p < +\infty \right\}$$

Theorem

If G is a super dense Polishable subgroup of $\mathbb{T}^{\mathbb{N}}$, then the relation E^G of **equivalence modulo G** *of elements of $\mathbb{T}^{\mathbb{N}}$ is not classifiable by countable structures.*

Non-classification of pure states

Proposition (Farah, 2010)

Suppose that A is a non type I C-algebra.*
The relation E^{ℓ_2} is reducible to the relation $E_{\mathcal{P}(A)}$ of unitary equivalence on the set of pure states $\mathcal{P}(A)$ of A.
In particular, the latter relation is not classifiable by countable structures.

Consider the case when A is the CAR algebra.
A reduction of E^{ℓ_2} to $E_{\mathcal{P}(A)}$ is given by

$$\begin{array}{rcl} \mathbb{T}^{\mathbb{N}} & \to & \mathcal{P}\left(\mathbb{M}_{2^\infty}\right) \\ (z_i)_{i\in\mathbb{N}} & \mapsto & \bigotimes_{i\in\mathbb{N}} \omega_{z_i}, \end{array}$$

where ω_{z_i} is the vector state on $\mathbb{M}_2$ associated to the vector $(z_i, \bar{z}_i)$.

Non-classification of automorphisms

Proposition (L., 2012)

Suppose that $A = C \otimes \bigotimes_{n\in\mathbb{N}} B_n$, where $\mathbb{M}_2 \subset B_n$.
The relation E^{ℓ_2} is reducible to the relation $E^{u.e.}_{\mathrm{Aut}(A)}$ of unitary equivalence on the set $\mathrm{Aut}\,(A)$ of automorphisms of A.
In particular, the latter relation is not classifiable by countable structures.

Consider the case when A is the CAR algebra.
A reduction of E^{ℓ_2} to $E^{u.e.}_{\mathrm{Aut}(\mathbb{M}_{2^\infty})}$ is given by

$$\begin{aligned} \mathbb{T}^{\mathbb{N}} &\to \mathrm{Aut}\,(A) \\ (z_i)_{i\in\mathbb{N}} &\mapsto \bigotimes_{i\in\mathbb{N}} Ad\,(u_{z_i})\,, \end{aligned}$$

where u_{z_i} is a unitary in $\mathbb{M}_2$ with eigenvalues $z_i, \bar{z}_i$.

Non-classification tools

Tools to prove that some relation E is not classifiable (in some sense):

- **reduction** from a relation already known to be non-classifiable to E;
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Proper generic ergodicity

Consider a continuous action of a Polish group on a Polish space.

The action is called properly generically ergodic if

- all the **orbits** are **meager**;
- there is a **dense orbit**.

The importance of this notion is due to the following

Theorem

The orbit equivalence relation induced by a **properly generically ergodic** *continuous action is not smooth.*

Conjugation action

Consider a Polish group G and its action on itself by conjugation.

The orbits are in this case the **conjugacy classes**.

To prove meagerness of conjugacy classes, it is often useful the following

Criterion (Rosendal, 2007)

If, for every infinite subset P of $\mathbb{N}$, the set

$$\{g \in G \mid \exists n \in P,\ g^n = 1\}$$

is dense, then G has meager conjugacy classes.

Non-smoothness of conjugacy of automorphisms

Proposition

If A is one of the known examples of **strongly self absorbing C*-algebras**, *then the conjugation action of $\mathrm{Aut}(A)$ is properly generically ergodic.*
In particular, the relation of conjugacy of automorphisms of A is not smooth.

Consider a countable dense subset $\{\alpha_i\}_{i\in\mathbb{N}}$ of $\mathrm{Aut}(A)$ and an isomorphism $\psi : A \to A^{\otimes\mathbb{N}}$.

The element $\psi^{-1} \circ \bigotimes_{i\in\mathbb{N}} \alpha_i \circ \psi$ has **dense conjugacy class**.

Apply Rosendal's criterion to get **meagerness of orbits**.
Different argument for

- A real rank zero;
- $A = \mathcal{Z}$.

Non-classification tools

Tools to prove that some relation E is not classifiable (in some sense):

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Local orbits

Consider a cont. action of a Polish group G on a Polish space X.

Suppose that

- U is an open subset of X;
- $V = V^{-1}$ is an open neighborhood of 1 in G.

Consider the graph having

- the points of U as vertices;
- an arrow between x and y iff there is $g \in V$ such that $gx = y$.

If $x \in U$, the connected component of x in such graph is denoted by $\mathcal{O}(x, U, V)$ and called local orbit around x defined by U and V.

Generic turbulence

The point x is called turbulent iff for every open neighborhood U of x and open neighborhood $V = V^{-1}$ of 1 in G, $\mathcal{O}(x, U, V)$ is somewhere dense.

The action is called generically turbulent if

- there is a **turbulent point with dense orbit**;
- all the **orbits** are **meager**.

The importance of this notion is due to the following

Theorem (Hjorth, 2000)

The orbit equivalence relation induced by a generically turbulent continuous action is not classifiable by countable structures.

Proposition

If A is a **strongly self absorbing C*-algebra***, the action of $U(A)$ on $\mathrm{Aut}(A)$ is (generically) turbulent.*
In particular, the relation of **unitary equivalence** *on $\mathrm{Aut}(A)$ is not classifiable by countable structures.*

Since $\mathrm{Inn}(A)$ is a proper dense Borel subgroup of $\mathrm{Aut}(A)$, the orbits are dense and meager. Turbulence of points follows from the following

Lemma (Dadarlat-Winter, 2007)

For $F \subset A$ finite and $\varepsilon > 0$, define

$$N_{F,\varepsilon} = \{u \in U(A) \mid \forall x \in F,\ \|[x,u]\| < \varepsilon\}.$$

For every $F \subset A$ finite and $\varepsilon > 0$ there are $G \subset A$ finite and $\delta > 0$ such that for every $u \in N_{G,\delta}$ there is a path from 1 to u contained in $N_{F,\varepsilon}$.

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