Continuous logic and sofic groups

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July 20th, 2012

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Logic for metric structures

Logic for metric structures is a modification, or, better, a **generalization** of the usual first order logic.

It is a tool to study structures naturally endowed with a **metric** (complete and bounded).

Examples of such structures are:

- probability spaces;
- metric groups;
- (unit balls of) Banach algebras or C*-algebras.

Languages and structures

A (metric) language $\mathcal{L}$ is given by

- a set of **function symbols**;
- the **arity** $n_f \in \mathbb{N}$ of any function symbol f;
- a **continuity modulus** $\omega_f : \mathbb{R}_+ \to \mathbb{R}_+$ for any function symbol f.

A (metric) $\mathcal{L}$-structure $\mathcal{S}$ is given by

- a **complete metric space** S with metric bounded by 1, called **support** of $\mathcal{S}$;
- for every function symbol, a uniformly continuous function

$$f^{\mathcal{S}} : S^{n_f} \to S$$

with ω_f as uniform continuity modulus in every variable, called **interpretation** of f in $\mathcal{S}$.

Bi-invariant metric groups

If G is a group endowed with a bi-invariant metric,
left and right translation are isometric and, in particular, 1-Lipschitz.

Consider the language $\mathcal{L}_{Gr}$ containing:

- a binary function symbol "$\cdot$";
- a unary function symbol "$(\quad)^{-1}$";
- a zerary (or constant) function symbol "1"

having the identity function as continuity modulus.

A bi-invariant metric group is a structure in the language $\mathcal{L}_{Gr}$.

Discrete structures as metric structures

A **discrete group** can (and will be) regarded as a bi-invariant metric group endowed with the **trivial metric** defined by

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y; \\ 0 & \text{otherwise.} \end{cases}$$

More generally, a "discrete" structure of the usual first order logic can be regarded as a metric structure, endowed with the trivial metric.

Logic for metric structures can thus be regarded as a **generalization of usual first order logic**.

Terms

If $\mathcal{L}$ is a language, the $\mathcal{L}$-terms are defined, as in usual discrete logic, by recursion

- starting from **variable symbols** $x, y, z \ldots$ or constant symbols, and
- composing with the **function symbols** in $\mathcal{L}$.

The interpretation of an $\mathcal{L}$-term in a structure is defined as usual by recursion on its complexity.

For example,

$$\left(x \cdot \left(y \cdot \left(x^{-1} \cdot \left(y^{-1}\right)\right)\right)\right)$$

or, briefly,

$$[x, y]$$

is an $\mathcal{L}_{Gr}$-term,
whose interpretation in a bi-invariant metric group is **the function that associates to a pair of elements their commutator**.

A symbol for the metric

In logic for metric structures, **the extra-logic symbol "$=$" is missing**.

There is instead a symbol d, representing the metric.

d is interpreted in a structure $\mathcal{S}$ as the metric $d^{\mathcal{S}}$ of the support of $\mathcal{S}$.

Atomic formulae

The **atomic $\mathcal{L}$-formulae** have the form

$$d(t_1, t_2)$$

where t_1, t_2 are $\mathcal{L}$-terms.

The **interpretation** of an $\mathcal{L}$-formula in an $\mathcal{L}$-structure $\mathcal{S}$ is defined in the obvious way, interpreting d as the metric $d^{\mathcal{S}}$ of the support of $\mathcal{S}$.

For example

$$\varphi(x, y) \equiv d([x, y], 1)$$

is an atomic $\mathcal{L}_{Gr}$-formula.

The interpretation φ^G of φ in G is **the function associating to a pair of element the distance of their commutator from the identity**.

Formulae

The $\mathcal{L}$-formulae are defined by recursion

- starting from the **atomic formulae**;
- using the continuous functions $f : [0,1]^n \to [0,1]$ as **connectives**;
- using the operators $\sup_x$ and $\inf_x$ as **quantifiers**.

For example,

$$\psi \equiv \sup_x \sup_y d\left([x,y],1\right)$$

is an $\mathcal{L}_{Gr}$-formula.

Ultraproducts

Ultraproducts are defined as in discrete logic, but identifying two sequences when the distance of their values goes to zero.

Precisely, assume that

- $\mathcal{L}$ is a **language**;
- $(\mathcal{S}_n)_{n\in\mathbb{N}}$ is a **sequence** $\mathcal{L}$**-structures**;
- $\mathcal{U}$ is an **ultrafilter** on $\mathbb{N}$.

Define the pseudometric on $\prod_{n\in\mathbb{N}} S_n$

$$d^{\mathcal{U}}\left(\left((a_n)_{n\in\mathbb{N}},(b_n)_{n\in\mathbb{N}}\right)\right)=\mathcal{U}-\lim_{n\in\mathbb{N}} d^{\mathcal{S}_n}(a_n,b_n)$$

and denote by $S_{\mathcal{U}}$ the associated quotient complete metric space. The ultraproduct $\mathcal{S}_{\mathcal{U}}$ of $(\mathcal{S}_n)_{n\in\mathbb{N}}$ is the $\mathcal{L}$-structure having

- $S_{\mathcal{U}}$ as **support**;
- $f^{\mathcal{S}_{\mathcal{U}}}\left(\left[(a_n)_{n\in\mathbb{N}}\right]\right)=\left[\left(f^{\mathcal{S}_n}(a_n)\right)_{n\in\mathbb{N}}\right]$ as **interpretation** of f.

Łos' theorem

As in usual first order logic, ultraproducts satisfy the

Theorem (Łos)

If $\varphi(x_1, \ldots, x_k)$ is an $\mathcal{L}$-formula, then

$$\varphi^{\mathcal{S}_\mathcal{U}}\left(\left[\left(a_n^1\right)_{n\in\mathbb{N}}\right], \ldots, \left[\left(a_n^k\right)_{n\in\mathbb{N}}\right]\right) = \mathcal{U} - \lim_{n\in\mathbb{N}} \varphi^{\mathcal{S}_n}\left(a_n^1, \ldots, a_n^k\right).$$

In particular, if φ does not have free variables,

$$\varphi^{\mathcal{S}_\mathcal{U}} = \mathcal{U} - \lim_{n\in\mathbb{N}} \varphi^{\mathcal{S}_n}.$$

Moreover, ultraproducts are $\aleph_1$-saturated.

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Two classes of discrete groups

Sofic and **hyperlinear groups** are two classes of discrete groups intensively studied in geometric group theory.

Several **open problems** and conjectures in group theory have been **settled** in the case of sofic or hyperlinear groups.

Many problems remain open, among which:
is there a group which is not sofic (or hyperlinear)?

Even though sofic and hyperlinear groups are discrete groups, they are **defined in terms of metric groups**.

It is therefore natural to apply **logic for metric structures** to their study.

Two families of metric groups

Denote

- by $\mathfrak{S}_n$ the groups of permutations on n, endowed with the **Hamming metric**
$$d(\sigma,\tau)=\frac{1}{n}\left|\left\{i\in n \,|\, \sigma(i)\neq\tau(i)\right\}\right|;$$
- by U_n the groups of unitary $n\times n$ matrices, endowed with the **Hilbert-Schmidt metric**
$$d(A,B)=\frac{1}{2\sqrt{n}}\left(\sum_{i,j}\left|A_{ij}-B_{ij}\right|^2\right)^{\frac{1}{2}}$$
- by G a discrete group, endowed with the **trivial metric**
$$d(g,h)=1 \quad \text{se } g\neq h$$

Sofic groups

Interpret $\mathfrak{S}_n$, U_n and G as structures in the language $\mathcal{L}_{Gr}$.

Definition

G is called sofic if, for every $\varepsilon > 0$ and $F \subset G$ finite, there is, for some $n \in \mathbb{N}$, a function

$$f : G \to \mathfrak{S}_n$$

preserving on F all function symbols and the metric up to ε.

In formulae, this means that, $\forall g, h \in F$,

- $d^{\mathfrak{S}_n}\left(f\left(gh\right), f\left(g\right) f\left(h\right)\right) < \varepsilon$
- $d^{\mathfrak{S}_n}\left(f\left(g^{-1}\right), f\left(g\right)^{-1}\right) < \varepsilon$
- $d^{\mathfrak{S}_n}\left(f\left(1^G\right), 1^{\mathfrak{S}_n}\right) < \varepsilon$
- $\left|d^{\mathfrak{S}_n}\left(f\left(g\right), f\left(h\right)\right) - d^G\left(g, h\right)\right| < \varepsilon$

Brief history of sofic groups

Sofic groups where introduced Gromov in **1999**.

Since then, they have been subject of intensive study in group theory.

Several conjectures have been proved in the case of sofic groups.

Some examples:

Conjecture (Gottschalk)

If X is a finite discrete space and $f : X^G \to X^G$ is a continuous one-to-one function commuting with the Bernoulli action of G, then f is onto.

Conjecture (Kaplanski)

If K is a field, the group algebra $K(G)$ is directly finite, i.e. $ab = 1$ implies $ba = 1$.

A very wide class

The class of sofic groups includes in particular

- abelian groups:
- more generally, amenable groups (groups endowed with a finitely additive left invariant measure defined on all the subsets);
- even initially subamenable groups (groups whose finite subsets are isomorphic to finite subsets of amenable groups).

For example, **free groups** are sofic (but not amenable).
It is not known if hyperbolic groups are sofic.

Open problem

Is there a group which is not sofic?

It is conjectured that there is an (hyperbolic) non-sofic "monster group".

Hyperlinear groups

The definition of hyperlinear group is analogous to the definition of sofic group, where symmetric groups with the Hamming metric are replaced by **unitary groups with the Hilbert-Schmidt metric**.

Definition

G is called hyperlinear if, for every $\varepsilon > 0$ and $F \subset G$ finite, there is, for some $n \in \mathbb{N}$, a function

$$f : G \to U_n$$

preserving on F all function symbols and the metric up to ε.

Hyperlinear groups and sofic groups

The class of hyperlinear groups contains the class of sofic groups.

In fact, the function

$$\begin{array}{rcl} S_n & \to & U_n \\ \sigma & \mapsto & A_\sigma \end{array}$$

associating to a permutation the corresponding permutation matrix, is a group homomorphism compatible with the metric :

$$d\left(A_\sigma, A_\tau\right)^2 = \frac{1}{2} d\left(\sigma, \tau\right)$$

It is not known if there is any discrete group which is not hyperlinear.

Hyperlinear groups and Connes' embedding problem

Hyperlinear groups were introduced by Radulescu in **2000**, in relation with the so called "Connes embedding problem".

Connes embedding problem asks if any separable II_1 factor embeds in one (or any) **ultrapower** $R^{\mathcal{U}}$ **of the hyperfinite** II_1 **factor** R.

Theorem (Radulescu, 2000)

If G is a countable group, TFAE

1. *G is hyperlinear;*
2. *the group von Neumann algebra $L(G)$ is embeddable in $R^{\mathcal{U}}$.*

Thus, the existence of a non-hyperlinear countable group would imply a negative answer to Connes' embedding problem.

Ultraproducts of permutation groups

If $\mathcal{U}$ is a nonprincipal **ultrafilter** over $\mathbb{N}$, the corresponding ultraproduct $\mathfrak{S}_{\mathcal{U}}$ of the sequence $(\mathfrak{S}_n)_{n\in\mathbb{N}}$ is a bi-invariant metric groups, having

- $\prod_n \mathfrak{S}_n / \sim_{\mathcal{U}}$ as **support**, where

$$(\sigma_n) \sim_{\mathcal{U}} (\tau_n) \quad \text{sse} \quad \mathcal{U} - \lim_{n\in\mathbb{N}} d(\sigma_n, \tau_n)$$

- **operation**

$$[(\sigma_n)] \cdot [(\tau_n)] = [(\sigma_n \tau_n)]$$

- **"ultralimit" metric**

$$d^{\mathfrak{S}_{\mathcal{U}}}([(\sigma_n)], [(\tau_n)]) = \mathcal{U} - \lim_n d^{\mathfrak{S}_n}(\sigma_n, \tau_n).$$

Universal sofic group

Recall the

Definition

G is called sofic if, for every $\varepsilon > 0$ and $F \subset G$ finite, there is, for some $n \in \mathbb{N}$, a function

$$f : G \to \mathfrak{S}_n$$

preserving on F all function symbols and the metric up to ε.

It is easy to see that the following statements are equivalent for a countable discrete group G:

- G is **sofic**;
- G is **(isometrically) isomorphic** to a subgroup of $\mathfrak{S}_{\mathcal{U}}$ for every $\mathcal{U}$;
- G is **(isometrically) isomorphic** to a subgroup of $\mathfrak{S}_{\mathcal{U}}$ for some $\mathcal{U}$.

In view of this, the groups $\mathfrak{S}_{\mathcal{U}}$ are called universal sofic groups.

Universal hyperlinear groups

The hyperlinear case is analogous:

Definition

G is called hyperlinear if, for every $\varepsilon > 0$ and $F \subset G$ finite, there is, for some $n \in \mathbb{N}$, a function

$$f : G \to U_n$$

preserving on F all function symbols and the metric up to ε.

It is easily seen that the following statements are equivalent for a countable discrete groups G:

- G is **hyperlinear**;
- G is **(isometrically) isomorphic** to a subgroup of $U_{\mathcal{U}}$ for every $\mathcal{U}$;
- G is **(isometrically) isomorphic** to a subgroup of $U_{\mathcal{U}}$ for some $\mathcal{U}$.

In view of this, the groups $U_{\mathcal{U}}$ are called universal hyperlinear groups.

The isomorphism problem

Question

Are all universal sofic (respectively hyperlinear) groups isomorphic, algebraically or as metric groups?

These groups were considered as quotient of discrete ultraproducts.

This was an obstacle to the application of methods from model theory.

Simon Thomas proved using “discrete” logic combined with an algebraic argument the following:

Theorem (Thomas, 2010)

If CH fails, there are $2^{\mathfrak{c}}$ not algebraically isomorphic universal sofic groups.

It is not clear how one can adapt Thomas’ proof to the **hyperlinear case**.

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The order property

The point of view of continuous logic allows one to apply techniques from model theory, such as the criterion known as order property.

Assume in the following that

- $\mathcal{L}$ is a **language**;
- $(\mathcal{S}_n)_{n\in\mathbb{N}}$ is a **sequence of separable $\mathcal{L}$-structures**;
- $\varphi(x_1,\ldots,x_k,y_1,\ldots,y_k)$ is a **formula** in $2k$ free variables.

If $n\in\mathbb{N}$, define the k-ary relation $\prec_\varphi$ on S_n

$$(a_1,\ldots,a_k)\prec_\varphi(b_1,\ldots,b_k)$$

iff

$$\varphi^{\mathcal{S}_n}(a_1,\ldots,a_k,b_1,\ldots,b_k)=0$$

e

$$\varphi^{\mathcal{S}_n}(b_1,\ldots,b_k,a_1,\ldots,a_k)=1$$

The order property

Say that the sequence $(\mathcal{S}_n)_{n\in\mathbb{N}}$ has the **order property** witnessed by φ if the structures $\mathcal{S}_n$ contain arbitrarily long $\prec_\varphi$-chains.

More precisely, for every $l \in \mathbb{N}$ there is a structure $\mathcal{S}_n$ containing a $\prec_\varphi$-chain of length l.

The importance of the order property is due to the following

Theorem (Farah-Shelah, 2009)

Assume that $(\mathcal{S}_n)_{n\in\mathbb{N}}$ has the order property. If CH fails, then there are $2^{\mathfrak{c}}$ many pairwise non-isomorphic ultraproducts $\mathcal{S}_{\mathcal{U}}$.

The sequences $(\mathfrak{S}_n)_{n\in\mathbb{N}}$ and $(U_n)_{n\in\mathbb{N}}$ have the order property.

Suppose that $l \in \mathbb{N}$.
Consider the embedding

$$\begin{aligned} \overbrace{\mathfrak{S}_3 \times \cdots \times \mathfrak{S}_3}^{l \text{ times}} &\to \mathfrak{S}_{3^l} \\ (\rho_1, \ldots, \rho_l) &\mapsto \rho_1 \otimes \cdots \otimes \rho_l \end{aligned}$$

given by the coordinate-wise action of $\overbrace{\mathfrak{S}_3 \times \cdots \times \mathfrak{S}_3}^{l \text{ times}}$ on $\overbrace{3 \times 3 \times \ldots \times 3}^{l \text{ times}}$.
Define, for $i, j \in \{1, 2, \ldots, l\}$,

$$\sigma_i = \overbrace{(01) \otimes \cdots \otimes (01)}^{i \text{ times}} \otimes \overbrace{e^{\mathfrak{S}_3} \otimes \cdots \otimes e^{\mathfrak{S}_3}}^{l-i \text{ times}}$$

and

$$\tau_i = \overbrace{e^{\mathfrak{S}_3} \otimes \cdots \otimes e^{\mathfrak{S}_3}}^{j-1 \text{ times}} \otimes (12) \otimes \overbrace{e^{\mathfrak{S}_3} \otimes \cdots \otimes e^{\mathfrak{S}_3}}^{l-j \text{ times}}$$

Observe that

- if $i < j$, then $[\sigma_i, \tau_j] = e^{\mathfrak{S}_{3^l}}$ and $d\left([\sigma_i, \tau_j], e^{\mathfrak{S}_{3^l}}\right) = 0$;
- se $i \geq j$ then

$$[\sigma_i, \tau_j] = \overbrace{e^{\mathfrak{S}_3} \otimes \cdots \otimes e^{\mathfrak{S}_3}}^{j-1 \text{ times}} \otimes (012) \otimes \overbrace{e^{\mathfrak{S}_3} \otimes \cdots \otimes e^{\mathfrak{S}_3}}^{l-j \text{ times}}$$

and hence $d\left([\sigma_i, \tau_j], e^{\mathfrak{S}_{3^l}}\right) = 1$

This proves that

$$(\sigma_i, \tau_i)_{i=1}^{l}$$

is a $\prec_\varphi$-chain of length l, where $\varphi(x_1, x_2, y_1, y_2)$ is the formula

$$d([x_1, y_2], 1)$$

Thus, φ witnesses the order property of the sequence $(\mathfrak{S}_n)_{n\in\mathbb{N}}$.

Also the sequence $(U_n)_{n\in\mathbb{N}}$ has the order property.

The order property for $(U_n)_{n\in\mathbb{N}}$ can be deduced from the order property $(\mathfrak{S}_n)_{n\in\mathbb{N}}$ using the homomorphic embedding

$$\begin{array}{rcl} \mathfrak{S}_n & \to & U_n \\ \sigma & \mapsto & A_\sigma \end{array}$$

via permutation matrices.

In view of the relation

$$d\left(A_\sigma, A_\tau\right) = \sqrt{\frac{d\left(\sigma,\tau\right)}{2}}.$$

this embedding almost preserves the value of quantifier-free formulae.

We are not done yet...

This shows that, if CH fails, there are $2^{\mathfrak{c}}$ universal sofic (resp. hyperlinear) groups that are pairwise non-isomorphic as metric groups.

More precisely, between any two of them there is no isometry preserving the operation.

This does not necessarily implies that between them there is no bijection **preserving the operation**.

To obtain this stronger result, one has to analyze **the proof of Farah-Shelah's theorem**.

Suppose that

- $\mathcal{L}$ is a **language**;
- $(\mathcal{S}_n)_{n\in\mathbb{N}}$ is a **sequence of $\mathcal{L}$-structures** with the order property witnessed by $\varphi(x_1, \ldots, x_k, y_1, \ldots, y_k)$.

Skeleton like chains

A $\prec_\varphi$-chain $\mathcal{C}$ in an $\mathcal{L}$-structure $\mathcal{S}$ is called **$(\aleph_1, \varphi)$-skeleton like** if for every $\overline{a} = (a_1, \ldots, a_k) \in S^k$ there is $\mathcal{C}_{\overline{a}} \subset \mathcal{C}$ countable such that, for every $\overline{b}, \overline{c} \in \mathcal{C}$, if

$$\left\{x \in \mathcal{C}_{\overline{a}} \,\middle|\, \overline{b} \prec_\varphi x \prec_\varphi \overline{c}\right\} = \varnothing,$$

then

$$\varphi^{\mathcal{S}}\left(\overline{a}, \overline{b}\right) = \varphi^{\mathcal{S}}\left(\overline{a}, \overline{c}\right) \quad \text{and} \quad \varphi^{\mathcal{S}}\left(\overline{b}, \overline{a}\right) = \varphi^{\mathcal{S}}\left(\overline{c}, \overline{a}\right).$$

Lemma (1)

If I is a linear order of cardinality $\mathfrak{c}$, then there is an ultrafilter $\mathcal{U}$ over $\mathbb{N}$ such that $\mathcal{S}_{\mathcal{U}}$ contains a $(\aleph_1, \varphi)$-skeleton like $\prec_\varphi$-chain of order type I.

The lemma is obtained showing that the collection of subsets of $\mathbb{N}$ that ensure that $\mathcal{U}$ satisfies the thesis has the f.i.p.

Non-isomorphic structures

Lemma (2)

Suppose that $\lambda \geq \aleph_2$ is a cardinal. If **K** *is a class of $\mathcal{L}$-structures of character density λ such that, for every linear order I of cardinality λ, there is a structure $\mathcal{S}$ in* **K** *containing an $(\aleph_1, \varphi)$-skeleton like $\prec_\varphi$-chain of order type I, then* **K** *contains 2^λ many pairwise non-isomorphic structures.*

In the proof, one associates invariants to every **linear order** I of cardinality λ and every $\mathcal{L}$-**structure** of character density λ, in such a way that

- the number of invariants associated to linear orders is 2^λ;
- at most λ invariants are associated to a structure;
- is a structure $\mathcal{S}$ contains a $(\aleph_1, \varphi)$-skeleton like $\prec_\varphi$-chain $\mathcal{C}$, then the set of invariants associated to $\mathcal{S}$ contains the set of invariants associated to $\mathcal{C}$.

The conclusion follows by the (infinite) pigeon-hole principle.

$2^{\mathfrak{c}}$ universal sofic groups

Let φ the formulae witnessing the **order property** of the sequence $(\mathfrak{S}_n)_{n\in\mathbb{N}}$ with the **Hamming metric**.

Apply **Lemma 1** to the sequence $(\mathfrak{S}_n)_{n\in\mathbb{N}}$.

This gives, for every linear order I of cardinality $\mathfrak{c}$, an **ultraproduct** $\mathfrak{S}_{\mathcal{U}}$ endowed with the **ultralimit metric** containing a $(\aleph_1, \varphi)$-skeleton like $\prec_\varphi$-chain of order type I.

Observe now that $\mathcal{C}$ remains a $(\aleph_1, \varphi)$-skeleton like $\prec_\varphi$-chain if the ultralimit metric is replaced by the trivial metric.

If $\mathfrak{c} \geq \aleph_2$, applying **Lemma 2** to the class of ultraproducts $\mathfrak{S}_{\mathcal{U}}$ endowed with the trivial metric, one infers the existence of $2^{\mathfrak{c}}$ pairwise algebraically non-isomorphic sofic groups.

$2^{\mathfrak{c}}$ universal hyperlinear groups

Reasoning in the same way in the case of $(U_n)_{n\in\mathbb{N}}$, one obtains, if $\mathfrak{c} \geq \aleph_2$, $2^{\mathfrak{c}}$ pairwise algebraically non-isomorphic universal hyperlinear groups.

More generally, one obtains the following

Lemma

Let $\mathcal{L}$ be a **language** *and φ be an $\mathcal{L}$-***formula** *of the form*

$$q\left(d\left(t, s\right)\right)$$

where q is a **continuous function** *such that $q(x) = 0$ iff $x = 0$.*

Suppose that $(\mathcal{S}_n)_{n\in\mathbb{N}}$ is a sequence of $\mathcal{L}$-structures with the order property witnessed by φ.
If CH fails, then there are $2^{\mathfrak{c}}$ many ultraproducts $\mathcal{S}_{\mathcal{U}}$ such that there are no bijections preserving the function symbols between any two of them.

Isomorphic universal sofic groups

If CH holds, the ultraproducts $\mathfrak{S}_\mathcal{U}$ are saturated.

As in usual first order logic, if $\mathcal{S}, \mathcal{S}'$ are saturated structures, TFAE

- $\mathcal{S}$ and $\mathcal{S}'$ are isomorphic;
- $\mathcal{S}$ and $\mathcal{S}'$ are elementarily equivalent: $\varphi^\mathcal{S} = \varphi^{\mathcal{S}'}$ for every sentence φ.

Thus, if CH holds, all the universal sofic groups are **isometrically isomorphic** iff they are all **elementarily equivalent**.

Łos theorem gives the equivalence of the following statements:

- all the universal groups $\mathfrak{S}_\mathcal{U}$ are **elementarily equivalent**;
- for every $\mathcal{L}_{Gr}$-sentence φ, the sequence $\left(\varphi^{\mathfrak{S}_n}\right)_{n\in\mathbb{N}}$ is convergent.

The same holds for universal hyperlinear groups.

Convergence of theories

Proposition (L., 2011)

If φ is a $\mathcal{L}_{Gr}$-sentence with alternation of at most two quantifiers, then the sequences $\left(\varphi^{\mathfrak{S}_n}\right)_{n\in\mathbb{N}}$ and $\left(\varphi^{U_n}\right)_{n\in\mathbb{N}}$ are convergent.

Question

Is it true **for all $\mathcal{L}_{Gr}$-sentences** *φ that the sequence $\left(\varphi^{\mathfrak{S}_n}\right)_{n\in\mathbb{N}}$ is convergent?*

A positive answer to this question would imply that, if CH holds, all universal sofic groups are isometrically isomorphic.

The same holds for unitary groups and universal hyperlinear groups.

You can find these slides on my website **lupini.org**.

Thank you for the attention!