# Non-classifiability of automorphisms of C*-algebras up to unitary equivalence

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1. Complexity of classification problems
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# Equivalence relations

Descriptive set theory offers a framework where one can study and compare the **complexity** of different equivalent relations $(X, E)$.

In this setting,

- the set $X$ of objects is supposed to be endowed with a **Polish topology**, or at least a standard Borel structure;
- the relation $E \subset X \times X$ is **Borel** or at least analytic.

For example, suppose that

- $X$ is the set $\mathrm{Aut}\,(A)$ of **automorphisms of a C*-algebra** $A$;
- $E$ is the relation of **unitary equivalence**.

$\mathrm{Aut}\,(A)$ is a Polish space with respect to the **point-norm topology** , and $E$ is a Borel equivalence relation on $\mathrm{Aut}\,(A)$.

# Borel reduction

A **notion of comparison** between different equivalence relations is given by (Borel) reduction.

## Definition

A (Borel) reduction from $(X, E)$ to $(X', E')$ is a function $f : X \to X'$ s.t.

- $f$ is Borel;
- for $x, y \in X$, $xEy$ iff $f(x)\, E' f(y)$.

If there is a (Borel) reduction from $(X, E)$ to $(X', E')$, we say that $E$ is **(Borel) reducible** to $E'$, and we write

$$E \leq E'.$$

# Benchmark equivalence relations

These are benchmark relations to be used as **measures of complexity:**

1. the relation $=_{\mathbb{R}}$ of **equality of real numbers**;
2. the **isomorphism** relation $\simeq_{\mathcal{C}}$ in a class $\mathcal{C}$ of **countable structures** (such as groups, rings etc...).

These (families of) relations give a hierarchy of complexity, since

$$=_{\mathbb{R}} \leq \simeq_{\mathcal{C}}$$

## Definition

An equivalence relation $E$ is called

1. **smooth** if it is reducible to $=_{\mathbb{R}}$;
2. **classifiable by countable structures** if it is reducible to $\simeq_{\mathcal{C}}$ for some class $\mathcal{C}$ of countable structures.

# Glimm's theorem

Suppose that

- $A$ is a separable C*-algebra;
- $\mathcal{H}$ is a separable Hilbert space;
- $\mathrm{Irr}_{\mathcal{H}}(A)$ is the Polish space of irreducible representations of $A$ on $\mathcal{H}$, endowed with the **topology of pointwise norm convergence**;
- $\sim^{A}_{u.e.}$ is the relation of unitary equivalence on $\mathrm{Irr}_{\mathcal{H}}(A)$, namely

$$\pi \sim^{A}_{u.e.} \pi' \qquad \text{iff} \qquad \pi' = Ad(u) \circ \pi \text{ for some } u \in U(\mathcal{H})$$

## Theorem (Glimm, 1961)

*For a separable C*-algebra A, TFAE*

1. *A is* **type I**
2. *the relation* $\sim^{A}_{u.e.}$ *on* $\mathrm{Irr}_{\mathcal{H}}(A)$ *is* **smooth**

# Non-classification by countable structures

Glimm's result was refined by Kerr-Li-Pichot and, independently, Farah.

**Theorem (2009)**

*If $A$ is a separable* **non type I** *C*-algebra, then the relation $\sim_{u.e.}$ is* **not classifiable by countable structures**.

The relation $E_{\ell_2}$ of $\ell_2$-equivalence in $(0,1)^{\mathbb{N}}$ is not classifiable.
If $\mathbb{M}_{2^\infty}$ is the CAR algebra and $A$ is any non type I C*-algebra, then

$$E_{\ell_2} \leq \sim_{u.e.}^{\mathbb{M}_{2^\infty}} \leq \sim_{u.e.}^{A}$$

The first reduction is given by

$$\begin{array}{rcl} (0,1)^{\mathbb{N}} & \to & \mathcal{P}\left(\mathbb{M}_{2^\infty}\right) \\ (t_n)_{n\in\mathbb{N}} & \mapsto & \bigotimes_{n\in\mathbb{N}} \omega_{t_n}, \end{array}$$

where $\omega_{t_i}$ is the vector state on $\mathbb{M}_2$ associated to the vector $(cos\, t_n, sin\, t_n)$

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# Phillips' theorem

Suppose that

- $A$ is a separable unital C*-algebra;
- $U(A)$ is the unitary group of $A$;
- $\mathrm{Aut}(A)$ is the automorphism group of $A$ endowed with the **point-norm topology**;
- $\approx^A_{u.e.}$ is the relation of unitary equivalence on $\mathrm{Aut}(A)$, namely

$$\alpha \approx^A_{u.e.} \beta \qquad \text{iff} \qquad \alpha = \mathrm{Ad}(u) \circ \beta \text{ for some } u \in U(A)$$

**Theorem (J. Phillips, 1985)**

*If $A$ is a separable unital C*-algebra, TFAE*

1. *$A$ has* **continuous trace***;*
2. *$\approx^A_{u.e.}$ is* **smooth***.*

# A dichotomy conjecture

## Conjecture

*If $A$ is a separable unital C*-algebra which does not have continuous trace, then $\approx^A_{u.e.}$ is* **not classifiable by countable structures**.

*In fact, even the approximately inner centrally trivial automorphisms of $A$ are* **not classifiable up to unitary equivalence by countable structures**.

Goal: verify this conjecture for most (all?) separable unital C*-algebras that do not have continuous trace.

In the following, assume that $A$ is a separable unital C*-algebra.

## The UHF case

Suppose that $A = \mathbb{M}_2^{\otimes \mathbb{N}}$ is the **CAR algebra**.

Consider the Borel function

$$\begin{array}{rccc} F: & (0,1)^{\mathbb{N}} & \to & \mathrm{Aut}(A) \\ & (t_n)_{n\in\mathbb{N}} & \mapsto & \bigotimes_{n\in\mathbb{N}} \mathrm{Ad}(u_{t_n}), \end{array}$$

where

$$u_{t_n} = \begin{pmatrix} \cos t_n & \sin t_n \\ -\sin t_n & \cos t_n \end{pmatrix}.$$

It turns out that $F$ is a **Borel reduction** of $E_{\ell_2}$ to $\approx^{A}_{u.e.}$.

Modifying $F$, one can get automorphisms with the **Rokhlin property**.

A similar reduction can be defined for **UHF algebras**.

# Hjorth's criterion

## Criterion (Hjorth, 2000)

*If there is a Borel function*

$$F : \mathbb{R}^{\mathbb{N}} \to \mathrm{Aut}(A)$$

*such that*

1. $\forall x \in \mathbb{R}^{\mathbb{N}}$, $\forall z \in \ell_p$,
$$F(x+z) \approx^{A}_{u.e.} F(x);$$
2. *for every comeager* $C \subset \mathbb{R}^{\mathbb{N}}$ *there are* $x, y \in C$ *such that*
$$F(x) \not\approx^{A}_{u.e.} F(y);$$

*then* $\approx^{A}_{u.c}$ *is* **not classifiable by countable structures**.

# The $\mathcal{Z}$-stable case

Suppose that $A \simeq B \otimes C^{\otimes \mathbb{N}}$ where

- $B, C$ are C*-algebras;
- $C$ is **non-abelian**.

For example, if $A$ is $\mathcal{Z}$-**stable**, $A \simeq A \otimes \mathcal{Z}^{\mathbb{N}}$.

Fix an element $a \in C_{sa}$ such that $\exp(ia)$ is not in the center.

Define the **continuous group homomorphism**

$$\begin{array}{rccc} F: & \mathbb{R}^{\mathbb{N}} & \to & \mathrm{Aut}\left(B \otimes C^{\otimes \mathbb{N}}\right) \\ & (t_n)_{n\in\mathbb{N}} & \mapsto & id_B \otimes \bigotimes_{n\in\mathbb{N}} \mathrm{Ad}\left(\exp\left(it_n a\right)\right) \end{array}$$

Replacing $id_B$ with $\alpha$ with the **(weak/tracial) Rohlin property**, one obtains automorphisms with the same property in the range of $F$.

Observe that

- $F[\ell_1] \subset \mathrm{Inn}(A)$;
- $F(\mathbf{1}) \notin \mathrm{Inn}(A)$, where $\mathbf{1}$ is the sequence constantly equal to 1.

It follows that,

1. if $x \in \mathbb{R}^{\mathbb{N}}$, $z \in \ell_1$,

$$F(x+z) = F(x)\,F(z) \approx_{u.e.} F(x)\,;$$

2. $F^{-1}[\mathrm{Inn}(A)]$ is a **proper dense Borel (hence meager) subgroup**;
If $X \subset \mathbb{R}^{\mathbb{N}}$ is such that, $\forall x, y \in X$,

$$F(x) \approx_{u.e.} F(y)\,,$$

then $X$ is contained in a coset of $F^{-1}[\mathrm{Inn}(A)]$ and it is meager.

# A closer look at the proof

Define

$$x_n = 1_B \otimes \overbrace{1_C \otimes \ldots \otimes 1_C}^{n-1 \text{ times}} \otimes a \otimes 1_C \otimes \ldots \in U\left(B \otimes C^{\otimes \mathbb{N}}\right).$$

Observe that

$$F\left((t_n)_{n\in\mathbb{N}}\right) = id_B \otimes \bigotimes_{n\in\mathbb{N}} \mathrm{Ad}\left(\exp\left(it_n a\right)\right) = \lim_{n\in\mathbb{N}} \mathrm{Ad}\left(\prod_{k<n} \exp\left(it_k x_k\right)\right).$$

- $(x_n)_{n\in\mathbb{N}}$ pairwise commuting $\Rightarrow$ $F$ is a homomorphism;
- $(x_n)_{n\in\mathbb{N}}$ is central $\Rightarrow$ $\lim_{n\in\mathbb{N}} \mathrm{Ad}\left(\prod_{k<n} \exp\left(it_k x_k\right)\right)$ is **well defined**;
- $(x_n)_{n\in\mathbb{N}}$ is "nontrivial" $\Rightarrow$ $\lim_{n\in\mathbb{N}} \mathrm{Ad}\left(\prod_{k<n} \exp\left(ix_k\right)\right)$ is **not inner**.

# Central sequences

A bounded sequence $(x_n)_{n\in\mathbb{N}}$ is called

- **central** (C) if $\lim_{n\in\mathbb{N}} \|[x_n, a]\| = 0$ for every $a \in A$;
- **hypercentral** (HC) if it is central, and $\lim_{n\in\mathbb{N}} \|[x_n, y_n]\| = 0$ for every central sequence $(y_n)_{n\in\mathbb{N}}$;
- **uniformly central** (UC) if $\lim_{n\in\mathbb{N}} \sup_{\|a\|\leq 1} \|[x_n, a]\| = 0$;
- **trivially central** (TC) if $\lim_{n\in\mathbb{N}} \|x_n - z_n\| = 0$ for some $z_n \in Z(A)$.

One has

$$\mathrm{TC} \subseteq \mathrm{UC} \subseteq \mathrm{HC} \subseteq \mathrm{C}$$

## Proposition (Elliott, Akemann-Pedersen)

*If $A$ is a separable C*-algebra, TFAE*

- $A$ **does not have continuous trace**;
- $TC \subsetneq C$.

# Three cases

Suppose that $A$ does not have continuous trace.
There are three (mutually exclusive) cases:

1. TC$\subseteq$UC$\subseteq$HC$\subsetneq$C;
2. TC$\subseteq$UC$\subsetneq$HC$=$C;
3. TC$\subsetneq$UC$=$HC$=$C.

Consider now Case 1.

## Proposition

*Suppose that $A$ has a* **central sequence which is not hypercentral***:*

$$TC \subseteq UC \subseteq HC \subsetneq C$$

*The automorphisms of $A$ are* **not classifiable up to unitary equivalence by countable structures***.*

# Automorphisms from central sequences

Suppose in the following

- $(y_n)_{n\in\mathbb{N}}$ is a central sequence in $A$;
- $y_n$ is positive;
- $\|y_n\| \leq \frac{1}{2}$.

## Lemma (1)

*There is a subsequence $(x_n)_{n\in\mathbb{N}}$ of $(y_n)_{n\in\mathbb{N}}$ such that the function*

$$\begin{array}{rccc} F: & (0,1)^{\mathbb{N}} & \to & \mathrm{Aut}(A) \\ & (t_n)_{n\in\mathbb{N}} & \mapsto & \lim_n \mathrm{Ad}\left(\prod_{k<n} \exp(it_k x_k)\right) \end{array}$$

- *is* **well defined**;
- *is* **continuous**;
- *sends $\ell_1$-equivalent sequences to unitarily equivalent automorphisms.*

# Outer automorphisms from non-trivial central sequences

## Lemma (2)

*Assume that, moreover, $(y_n)_{n\in\mathbb{N}}$ is not hypercentral.*
*One can choose $(x_n)_{n\in\mathbb{N}}$ so that, if $(t_n)_{n\in\mathbb{N}}, (t'_n)_{n\in\mathbb{N}} \in (0,1)^{\mathbb{N}}$ are such that $|t_n - t'_n| > \frac{1}{2}$ frequently, then the sequence*

$$\left( u \left( \prod_{k<n} \exp\left(it_k x_k\right) \right) \left( \prod_{k<n} \exp\left(it'_k x_k\right) \right)^* \right)_{n\in\mathbb{N}}$$

*is* **not central**,
*i.e.*

$$F\left((t_n)_{n\in\mathbb{N}}\right) \not\approx_{u.e.} F\left(\left(t'_n\right)_{n\in\mathbb{N}}\right).$$

# Nonclassification in Case 1

## Proposition

*Suppose that* $A$ *has a* **central sequence which is not hypercentral***:*

$$TC \subseteq UC \subseteq HC \subsetneq C$$

*The automorphisms of* $A$ *are* **not classifiable up to unitary equivalence by countable structures***.*

Observe that, if $C \subset (0,1)^{\mathbb{N}}$ is comeager,
it contains $(t_n)_{n\in\mathbb{N}}, (t'_n)_{n\in\mathbb{N}}$ such that $|t_n - t'_n| > \frac{1}{2}$ frequently.

Thus, if $(x_n)_{n\in\mathbb{N}}$ and $F$ are as in Lemma 2,
then $F$ satisfies the hypothesis of Horth's criterion:

1. $F$ sends $\ell_1$-equivalent sequences to unitarily equiv. automorphisms;
2. if $C \subset (0,1)^{\mathbb{N}}$ is comeager, $F$ sends at least two elements of $C$ to non-unitarily equivalent automorphisms.

# Nonclassification in Case 2

**Proposition**

*Suppose that A*

- **every central sequence of $A$ is hypercentral**, *and*
- **there is a (hyper)central sequence which is not uniformly central***:*

$$TC \subseteq UC \subsetneq HC = C$$

*The automorphisms of $A$ are* **not classifiable up to unitary equivalence by countable structures***.*

The proof is analogous to Case 1.

# Conclusion

Case 3: work in progress....

One can put together Case 1 and Case 2, obtaining the

**Theorem (L., 2012)**

*If A contains a central sequence which is not uniformly central, then the automorphisms of A are* **not classifiable up to unitary equivalence by countable structures**.

This covers most of the C*-algebras that do not have continuous trace, in particular the simple ones.