On the number of universal sofic and hyperlinear groups

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Non-isomorphic universal sofic and hyperlinear groups

Question

How many pairwise non (isometrically) isomorphic universal sofic groups $\mathfrak{S}_{\mathcal{U}}$ are there?
What about universal hyperlinear groups $U_{\mathcal{U}}$?

Answer

If CH fails, there are $2^{\mathfrak{c}}$ universal sofic groups $\mathfrak{S}_{\mathcal{U}}$ that are pairwise non-isomorphic as discrete groups.
The same holds for universal hyperlinear groups $U_{\mathcal{U}}$.

This result can be obtained via logic for metric structures.
Logic for metric structures provides a concrete criterion to check if a sequence of structures has, assuming the failure of CH, many non-isomorphic ultraproducts: the order property.

Suppose in the following that

- $(\mathcal{S}_n)_{n\in\mathbb{N}}$ is a sequence of $\mathcal{L}$-structures, and
- $\varphi(x_1,\ldots,x_k,y_1,\ldots,y_k)$ is an $\mathcal{L}$-formula.

For every $n\in\mathbb{N}$, define the relation $\prec_\varphi$ on S_n^k by

$$(a_1,\ldots,a_k) \prec_\varphi (b_1,\ldots,b_k)$$

iff

$$\varphi^{\mathcal{S}_n}(a_1,\ldots,a_k,b_1,\ldots,b_k) = 0$$

and

$$\varphi^{\mathcal{S}_n}(b_1,\ldots,b_k,a_1,\ldots,a_k) = 1$$

The order property

We say that $(\mathcal{S}_n)_{n\in\mathbb{N}}$ has the order property witnessed by the formula $\varphi(x_1,\ldots,x_k,y_1,\ldots,y_k)$ if for ever $l\in\mathbb{N}$ there is $n\in\mathbb{N}$ such that S_n^k contains a $\prec_\varphi$-chain of length l.

Theorem (Farah-Shelah, 2009)

Suppose that CH fails. If the sequence $(\mathcal{S}_n)_{n\in\mathbb{N}}$ has the order property, then there are $2^{\mathfrak{c}}$ many pairwise non isometrically isomorphic ultraproducts $\mathcal{S}_{\mathcal{U}}$.

I will now show that the sequence $(\mathfrak{S}_n)_{n\in\mathbb{N}}$ has the order property, witnessed by the formula

$$\varphi(x_1,y_1,x_2,y_2)\equiv d\left(x_1y_2x_1^{-1}y_2^{-1},e\right).$$

Suppose $l \in \mathbb{N}$.
Consider the embedding

$$\begin{array}{rcl}\overbrace{\mathfrak{S}_3 \times \cdots \times \mathfrak{S}_3}^{l \text{ times}} & \to & \mathfrak{S}_{3^l} \\ (\rho_1, \dots, \rho_l) & \mapsto & \rho_1 \otimes \cdots \otimes \rho_l\end{array}$$

given by the action of $\overbrace{\mathfrak{S}_3 \times \cdots \times \mathfrak{S}_3}^{l \text{ times}}$ on $\{1,2,3\}^l$ coordinate-wise.
Define, for $i, j \in \{1, 2, .., l\}$,

$$\sigma_i = \overbrace{(12) \otimes \cdots \otimes (12)}^{i \text{ times}} \otimes \overbrace{e^{\mathfrak{S}_3} \otimes \cdots \otimes e^{\mathfrak{S}_3}}^{l-i \text{ times}}$$

and

$$\tau_i = \overbrace{e^{\mathfrak{S}_3} \otimes \cdots \otimes e^{\mathfrak{S}_3}}^{j-1 \text{ times}} \otimes (23) \otimes \overbrace{e^{\mathfrak{S}_3} \otimes \cdots \otimes e^{\mathfrak{S}_3}}^{l-j \text{ times}}$$

Observe that

- if $i < j$, $[\sigma_i, \tau_j] = e^{\mathfrak{S}_{3^l}}$;
- if $i \geq j$ then

$$[\sigma_i, \tau_j] = \overbrace{e^{\mathfrak{S}_3} \otimes \cdots \otimes e^{\mathfrak{S}_3}}^{j-1 \text{ times}} \otimes (123) \otimes \overbrace{e^{\mathfrak{S}_3} \otimes \cdots \otimes e^{\mathfrak{S}_3}}^{l-j \text{ times}}$$

  is the product of 3^{l-1} disjoint 3-cycles.

This shows that

$$(\sigma_i, \tau_i)_{i=1}^{l}$$

is a $\prec_\varphi$-chain of length l.
Hence, the sequence $(\mathfrak{S}_n)_{n\in\mathbb{N}}$ has the order property, witnessed by the formula φ.

The sequence $(U_n)_{n\in\mathbb{N}}$ has the order property as well.

The order property for the sequence $(U_n)_{n\in\mathbb{N}}$ is deduced from the order property of $(\mathfrak{S}_n)_{n\in\mathbb{N}}$ using the embedding

$$\begin{array}{rcl} \mathfrak{S}_n & \to & U_n \\ \sigma & \mapsto & A_\sigma \end{array}$$

via permutation matrices.

It is easily seen that

$$d^{U_n}(A_\sigma, A_\tau) = \sqrt{\frac{d^{\mathfrak{S}_n}(\sigma,\tau)}{2}}.$$

As a consequence, this embedding almost preserves the value of quantifier-free formulas.

This is enough to deduce that, if CH fails, there are $2^{\mathfrak{c}}$ many universal sofic and hyperlinear groups that are **pairwise non-isometrically isomorphic**.

In order have them **non-isomorphic as discrete groups**, one needs the following

Lemma (L., 2011)

Suppose that CH fails. If the sequence $(\mathcal{S}_n)_{n\in\mathbb{N}}$ has the order property witnessed by a formula of the form

$$q\left(d\left(t,t'\right)\right),$$

where t, t' are terms and $q:[0,1]\to[0,1]$ is a continuous function vanishing at 0, then there are $2^{\mathfrak{c}}$ many ultraproducts $\mathcal{S}_{\mathcal{U}}$ that are pairwise non isomorphic **even if the metric is replaced with the trivial discrete metric**.

Applying the lemma to the sequences of symmetric groups, that have the order property witnessed by the formula

$$\varphi(x_1, y_1, x_2, y_2) \equiv d\left(x_1 y_2 x_1^{-1} y_2^{-1}, e\right),$$

one obtains the following

Theorem (L. 2011)

If CH fails, there are $2^{\mathfrak{c}}$ many universal sofic groups that are pairwise non-isomorphic as discrete groups. The same holds for universal hyperlinear groups.

The statement for universal sofic groups had already been proved by **Simon Thomas** in 2010 with **different, algebraic methods**.

For hyperlinear groups, alternatively, one can

- consider logic for non-necessarily bounded metric structures;
- show that the sequence $(\mathbb{M}_n(\mathbb{C}))_{n\in\mathbb{N}}$ regarded as tracial vN-algebras has the order property;
- apply Farah-Shelah to get, assuming $\neg$CH, $2^{\mathfrak{c}}$ non-isomorphic ultraproducts $\mathbb{M}_{\mathcal{U}}(\mathbb{C})$ (this is a result of Farah and Shelah);
- observe that the unitary group of $\mathbb{M}_{\mathcal{U}}(\mathbb{C})$ is the universal hyperlinear group $U_{\mathcal{U}}$;
- apply Diximier's theorem, asserting that non-isomorphic tracial vN-algebras have non-isomorphic (as discrete groups) unitary groups.

This was pointed out by Łukasz Grabowski.

Open problem

Assuming CH, are all universal sofic groups (isometrically) isomorphic to each other?
What about universal hyperlinear groups?

The model-theoretical notion of **elementary equivalence**, and the techinque known as **EF-games** provide a concrete way to (possibly) obtain such result.

Definition

Two $\mathcal{L}$-structures $\mathcal{S}$, $\tilde{\mathcal{S}}$ are said elementarily equivalent if for any $\mathcal{L}$-formula φ with no free variables, $\varphi^{\mathcal{S}} = \varphi^{\tilde{\mathcal{S}}}$.

Isomorphic structures are elementarily equivalent.

Theorem

The converse is true for countably saturated structures of density character $\aleph_1$.

Corollary

If CH holds, two universal sofic groups $\mathfrak{S}_{\mathcal{U}}$ *and* $\mathfrak{S}_{\mathcal{V}}$ *are isomorphic iff* $\varphi^{\mathfrak{S}_{\mathcal{U}}} = \varphi^{\mathfrak{S}_{\mathcal{V}}}$ *for any* $\mathcal{L}_{Gr}$*-formula* φ *with no free variables.*

Theorem (Łos on ultraproducts)

If $\varphi(x_1, \ldots, x_k)$ is an $\mathcal{L}$-formula, then

$$\varphi^{\mathcal{S}_{\mathcal{U}}}\left(\left[\left(a_n^1\right)_{n\in\mathbb{N}}\right], \ldots, \left[\left(a_n^k\right)_{n\in\mathbb{N}}\right]\right) = \mathcal{U} - \lim_{n\in\mathbb{N}} \varphi^{\mathcal{S}_n}\left(a_n^1, \ldots, a_n^k\right).$$

In particular, if φ has no free variables,

$$\varphi^{\mathcal{S}_{\mathcal{U}}} = \mathcal{U} - \lim_{n\in\mathbb{N}} \varphi^{\mathcal{S}_n}.$$

Proof.

By recursion on the complexity of the formula. □

Corollary

If CH holds, two universal sofic groups $\mathfrak{S}_{\mathcal{U}}$ and $\mathfrak{S}_{\mathcal{V}}$ are isomorphic iff, for every formula φ with no free variables,

$$\mathcal{U} - \lim_{n \in \mathbb{N}} \varphi^{\mathfrak{S}_n} = \mathcal{V} - \lim_{n \in \mathbb{N}} \varphi^{\mathfrak{S}_n}.$$

We say that **the theories of the permutation groups are convergent** if, for every formula φ with no free variables, **the sequence** $\left(\varphi^{\mathfrak{S}_n}\right)_{n \in \mathbb{N}}$ **is convergent**.

Corollary

If CH holds, all universal sofic groups $\mathfrak{S}_{\mathcal{U}}$ are isomorphic to each other iff the theories of permutation groups are convergent.

Open problem

Are the theories of permuation groups converging?
What about the theories of unitary groups?

A possible method to show this is via the so called **EF-games**.

Suppose that $\varepsilon > 0$, $n, m \in \mathbb{N}$ and $w_1, \ldots, w_t$ are words in the variables $x_1, \ldots, x_s$.

The EF game $\mathcal{G}^{\varepsilon}_{w_1,\ldots,w_t}(\mathfrak{S}_n, \mathfrak{S}_m)$ is played by two players: S(poiler) and D(uplicator).

In the i-th turn,

- S chooses and element a_i from $\mathfrak{S}_n$ if i is even, or an element b_i from $\mathfrak{S}_m$ if i is odd;
- D chooses an element b_i from $\mathfrak{S}_m$ if i is even, or an element a_i from $\mathfrak{S}_n$ if i is odd.

The game finishes at the s-th turn.
D wins iff

$$\left| d^{\mathfrak{S}_n}\left(w_i\left(a_1, \ldots, a_s\right), e^{\mathfrak{S}_n}\right) - d^{\mathfrak{S}_m}\left(w_i\left(b_1, \ldots, b_s\right), e^{\mathfrak{S}_m}\right) \right| < \varepsilon$$

for every $i \in \{1, 2, \ldots, t\}$.

Lemma

TFAE

1. *the theories of permutation groups converge;*
2. *for every $\varepsilon > 0$ and words $w_1, \ldots, w_t$ in the variables $x_1, \ldots, x_s$, there is n such that, for all k, D has a winning strategy for $\mathcal{G}^{\varepsilon}_{w_1,\ldots,w_t}(\mathfrak{S}_n, \mathfrak{S}_{kn})$.*

More precisely, if φ has the form

$$\inf_{x_1} \sup_{x_2} \ldots \inf_{x_{s-1}} \sup_{x_s} q\left(d\left(w_1, e\right), \ldots, d\left(w, e\right)\right)$$

where $q : [0,1]^k \to [0,1]$ is a continuous function, then TFAE

1. *the sequence $\left(\varphi^{\mathfrak{S}_n}\right)_{n\in\mathbb{N}}$ converges;*
2. *for every ε there is n such that, for all k, D has a winning strategy for $\mathcal{G}^{\varepsilon}_{w_1,\ldots,w_t}(\mathfrak{S}_n, \mathfrak{S}_{kn})$.*

The following proposition is obtained using the method of EF-games.

Proposition (L., 2011)

If φ is an $\mathcal{L}_{Gr}$-formula of the form

$$\inf_x \sup_y q\left(d\left(w_1, e\right), \ldots, d\left(w_t, e\right)\right),$$

where $w_1, \ldots, w_t$ are words in x, y, then the sequences $\left(\varphi^{\mathfrak{S}_n}\right)_{n\in\mathbb{N}}$ and $\left(\varphi^{U_n}\right)_{n\in\mathbb{N}}$ converge.

In order to establish the Proposition, one needs the following

Lemma (L.,2011)

For every $\delta > 0$, there is n such that, for every k and $\sigma \in \mathfrak{S}_{kn}$, there are $\tau \in \mathfrak{S}_n$ and an isometric embedding $\Phi : \mathfrak{S}_n \to \mathfrak{S}_{kn}$ such that

$$d^{\mathfrak{S}_{kn}}\left(\sigma, \Phi\left(\tau\right)\right) < \delta.$$

The proposition can now be proved, considering a suitable EF-game in two turns.

For $\delta > 0$, suppose n is as in the lemma. Consider the EF game in two turns between $\mathfrak{S}_n$ and $\mathfrak{S}_{kn}$.

Turn 1
- S chooses $\sigma_1 \in \mathfrak{S}_{kn}$;
- D replies with $\tau_1 \in \mathfrak{S}_n$ such that $d^{\mathfrak{S}_{kn}}(\sigma_1, \Phi(\tau_1)) < \delta$.

Turn 2
- S chooses $\tau_2 \in \mathfrak{S}_n$
- D replies with $\sigma_2 = \Phi(\tau_2)$.

Φ is an isometric embedding sending τ_1, τ_2 δ-close to σ_1, σ_2.

Choosing δ small enough, one can ensure that

$$\left| d^{\mathfrak{S}_c}\left(w_i(\sigma_1, \sigma_2), e^{\mathfrak{S}_{kn}}\right) - d^{\mathfrak{S}_n}\left(w_i(\tau_1, \tau_2), e^{\mathfrak{S}_n}\right) \right| < \varepsilon$$

for every i.