Some nonstandard techniques in combinatorics

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Loeb measure

Suppose that

- V is a $*$-**finite set**;
- Σ_V^0 is the ring of **internal subsets** of V;
- Σ_V is the external σ-algebra generated by Σ_V^0;
- μ_V^0 is the countably additive set function $A \mapsto st\left(\frac{|A|}{|V|}\right)$ on Σ_V^0.

Applying the **Carathéodory-Hahn extension theorem**, one obtains a probability measure μ_V on Σ_V extending μ_V^0: the Loeb measure on V.

Proposition

If $A \in \Sigma_V$ then there is $B \in \Sigma_V^0$ such that $\mu_V(A \triangle B) = 0$.

Conditional expectation

If

- $(X, \mathcal{B}, \mu)$ is a probability measure space,
- $\mathcal{A}$ is a sub-σ-algebra of $\mathcal{B}$;
- $g \in L^1(\mu)$

there is a unique $\mathcal{A}$-measurable element

$$\mathbb{E}[g|\mathcal{A}] \in L^1(\mu),$$

called **conditional expectation** of g on $\mathcal{A}$, s.t. for every $A \in \mathcal{A}$

$$\int_A \mathbb{E}[g|\mathcal{A}]\, d\mu = \int_A g d\mu.$$

This defines a positive linear function

$$\mathbb{E}[\cdot|\mathcal{A}] : L^1(\mathcal{B}, \mu) \to L^1\left(\mathcal{A}, \mu_{|\mathcal{A}}\right).$$

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Random partitions

Suppose that (V, E) is a **graph** and (X, Y) is a pair of sets of vertices.

Define

$$\delta(X, Y) = \frac{|E \cap (X \times Y)|}{|X|\,|Y|}$$

to be the **density of arrows** between X and Y.

We say that (X, Y) is ε-**pseudo-random** if, for every $A \subset X$ and $B \subset Y$ such that $|A| > \varepsilon\,|X|$ and $|B| > \varepsilon\,|Y|$, one has

$$|\delta(A, B) - \delta(X, Y)| < \varepsilon.$$

Suppose that

- (V, E) is a graph;
- $(V_i)_{i \in m}$ is a partition of V;
- R is the set of $(i, j) \in m \times m$ such that (V_i, V_j) is ε-**pseudo-random**.

Definition

The partition $(V_i)_{i \in m}$ is an ε-**regular** m-**partition** if

$$\sum_{(i,j) \in R} \frac{|V_i| \, |V_j|}{|V|^2} > (1 - \varepsilon) \, .$$

Theorem (Szemerédi's regularity lemma, 1978)

For every $\varepsilon > 0$ there is $C(\varepsilon) \in \mathbb{N}$ such that **every finite graph** *admits an* ε-**regular** m-**partition** *for some $m \leq C(\varepsilon)$.*

By transfer, Szemerédi's regularity lemma is equivalent to the following: Suppose that

- ε is a standard strictly positive real number;
- (V, E) is an internal $*$-finite graph.

There are

- a finite internal partition $(V_i)_{i \in m}$ of V;
- a subset R of $m \times m$ such that
  - for every $(i, j) \in R$ and for every internal $A \subset V_i$ and $B \subset V_j$ such that $|A| > \varepsilon |V_i|$ and $|B| > \varepsilon |V_j|$, $|\delta(A, B) - \delta(V_i, V_j)| < \varepsilon$
  - $\sum_{(i,j) \in R} \frac{|V_i||V_j|}{|V|^2} > 1 - \varepsilon$.

I will present now a proof of the nonstandard version of Szemerédi's regularity lemma.

Fix $\varepsilon > 0$ and an internal $*$-finite graph $\mathcal{G} = (V, E)$.

Consider

- the Loeb measure space (V, Σ_V, μ_V);
- the product measure space $(V \times V, \Sigma_V \otimes \Sigma_V, \mu_V \otimes \mu_V)$;
- the Loeb measure space $(V \times V, \Sigma_{V\times V}, \mu_{V\times V})$,

and observe that

- $\Sigma_V \otimes \Sigma_V$ is a sub-σ-algebra of $\Sigma_{V\times V}$;
- $\mu_V \otimes \mu_V$ is the restriction to $\Sigma_V \otimes \Sigma_V$ of $\mu_{V\times V}$.

Consider the conditional expectation

$$f := \mathbb{E}\left[1_E | \Sigma_V \otimes \Sigma_V\right]$$

where 1_E is the characteristic function of the set of vertices.

If $A \subset V$ and $B \subset V$ are internal such that $\frac{|A|}{|V|}$ and $\frac{|B|}{|V|}$ are not infinitesimals, then

$$\begin{aligned}
\int_{A\times B} f d\mu_{V\times V} &= \int_{A\times B} 1_E d\mu_{V\times V} \\
&= st\left(\frac{|E\cap(A\times B)|}{|V|^2}\right) \\
&= st\left(\frac{|E\cap(A\times B)|}{|A|\,|B|}\right) st\left(\frac{|A|\,|B|}{|V|^2}\right) \\
&= st\left(\delta\left(A,B\right)\right) st\left(\frac{|A|\,|B|}{|V|^2}\right).
\end{aligned}$$

Since f is positive and $\Sigma_V \otimes \Sigma_V$-measurable,
there is a positive $\Sigma_V \otimes \Sigma_V$-measurable step function g such that

$$g \leq f$$

and

$$\int (f - g) \, d \left(\mu_V \otimes \mu_V \right) < \eta.$$

Define

$$A = \left\{ f - g \geq \eta^{\frac{1}{2}} \right\}.$$

By **Markov's inequality**,

$$\left(\mu_V \otimes \mu_V \right) (A) < \eta^{\frac{1}{2}}.$$

Consider

- a disjoint union of rectangles P such that
  - $A \subset P \subset V \times V$
  - $(\mu_V \otimes \mu_V)(P) < \eta^{\frac{1}{2}}$
- a partition $(V_i)_{i \in m}$ of V in non-null sets such that both g and 1_P are constant on $V_i \times V_j$ for every $i, j \in m \times m$.

Denote by d_{ij} the constant value of g on $V_i \times V_j$.

If $V_i \times V_j$ is disjoint from P, then $V_i \times V_j$ is $2\eta^{\frac{1}{2}}$-pseudo-random.

In fact, if $V_i \times V_j$ is disjoint from P, it is disjoint from

$$A = \left\{ f - g \geq \eta^{\frac{1}{2}} \right\}.$$

Thus,

$$d_{ij} \leq f_{|(V_i \times V_j)} < d_{ij} + \eta^{\frac{1}{2}}.$$

If $A \subset V_i$ and $B \subset V_j$ are such that $|A| > \varepsilon |V_i|$ and $|B| > \varepsilon |V_j|$, integrating over $A \times B$ one gets

$$d_{i,j} st \left(\frac{|A|\,|B|}{|V|^2} \right) \leq st \left(\delta \left(A, B \right) \right) st \left(\frac{|A|\,|B|}{|V|^2} \right) < \left(d_{i,j} + \eta^{\frac{1}{2}} \right) st \left(\frac{|A|\,|B|}{|V|^2} \right).$$

Thus,

$$\left| \delta \left(A, B \right) - \delta \left(V_i, V_j \right) \right| \leq \left| \delta \left(A, B \right) - d_{i,j} \right| + \left| \delta \left(V_i, V_j \right) - d_{i,j} \right| < \eta^{\frac{1}{2}} + \eta^{\frac{1}{2}}.$$

Choose $\eta < \left(\frac{\varepsilon}{2}\right)^2$.

If R is the set of $(i, j) \in m \times m$ such that $V_i \times V_j$ is ε-pseudo-random,

$$\begin{aligned} st\left(\sum_{(i,j)\in R} \frac{|V_i|\,|V_j|}{|V|^2}\right) &= \mu_{V\times V}\left(\bigcup_{(i,j)\in R}(V_i \times V_j)\right) \\ &\geq \mu_{V\times V}\left((V \times V)\setminus P\right) \\ &> 1 - \eta^{\frac{1}{2}} \\ &\geq 1 - \varepsilon. \end{aligned}$$

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Amenable groups

Suppose that G is a discrete group.

If $\varepsilon > 0$ and $H, K \subset G$ are subsets, we say that H is (K, ε)-**invariant** if

$$\frac{|H \triangle gH|}{|H|} < \varepsilon$$

for every $g \in K$.

Definition

G is called amenable if $\forall \varepsilon > 0$, $\forall K \subset G$ finite, there is a (K, ε)-invariant finite subset of G.

Nonstandard characterization

By transfer, one gets from the definition of amenable group:

G is amenable iff there is $E \subset {}^*G$ internal $*$-finite such that, $\forall g \in G$,

$$\frac{|E \triangle gE|}{|E|} \approx 0$$

Define

$$\mathbb{I}(G) = \left\{ E \subset {}^*G \ * \text{-finite} \,\middle|\, \forall g \in G, \ \frac{|E \triangle gE|}{|E|} \approx 0 \right\}.$$

Thus, G is amenable as soon as $\mathbb{I}(G)$ is nonempty

Example

$\mathbb{Z}$ is amenable: $\mathbb{I}(\mathbb{Z})$ contains all the infinite intervals.

Følner filter

Suppose that G is a discrete group.

For $\varepsilon > 0$ and $K \subset G$ finite, define

$$F_{K,\varepsilon} = \{H \subset G \text{ finite} \,|\, H \text{ is } (K,\varepsilon)\text{-invariant}\}.$$

Thus, G is **amenable** if and only if all the $F_{K,\varepsilon}$ are **nonempty**.

In this case,

$$\mathcal{F}_G = \{F_{K,\varepsilon} \,|\, \varepsilon > 0,\ K \subset G \text{ finite}\}$$

is a filter basis on $\wp\left([G]^{<\aleph_0}\right)$.

Banach density

Suppose that

- G is amenable;
- A is a subset of G.

Define

$$d(A) = \limsup_{\mathcal{F}_G} \frac{|A \cap H|}{|H|}.$$

This is the so called **Banach density**.

By transfer, assuming enough saturation,

$$d(A) = \max \left\{ st\left(\frac{|{}^*A \cap E|}{|E|} \right) \mid E \in \mathbb{I}(G) \right\}.$$

Example

In $\mathbb{Z}$, d is the usual Banach density.

Correspondence principle

Theorem (Furstenberg's correspondence principle)

If G is an amenable group and $A \subset G$,
then there are

- *a probability measure space $(X, \mathcal{B}, \mu)$;*
- *a measure preserving near-action $(T_g)_{g \in G}$ of G on it;*
- *a measurable set $A_0 \in \mathcal{B}$ such that,*
  - $\mu(A_0) = d(A)$;
  - *for all finite subsets H of G,*

$$d\left(\bigcap_{g \in H} gA\right) \geq \mu\left(\bigcap_{g \in H} T^g [A_0]\right).$$

Proof:

Consider

- $E \in \mathbb{I}(G)$ such that $\frac{|{}^*A \cap E|}{|E|} \approx d(A)$;
- the Loeb measure space (E, Σ_E, μ_E);
- the near-action of G on it by left translation;
- and $A_0 = {}^*A \cap E$.

Thus,

$$\begin{aligned}
\mu\left(\bigcap_{g \in H} T^g [A_0]\right) &= st\left(\frac{\left|\bigcap_{g \in H} g({}^*A \cap E) \cap E\right|}{|E|}\right) \\
&= st\left(\frac{\left|{}^*\left(\bigcap_{g \in H} gA\right) \cap E\right|}{|E|}\right) \\
&\leq d\left(\bigcap_{g \in H} gA\right).
\end{aligned}$$