THE COHERENT CATEGORY OF TOWERS OF CHAIN COMPLEXES

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Abstract. We introduce a coherent category of towers of chain complexes in a given additive category, and describe the homotopy limit functor from such a category to the homotopy category of chain complexes. This category can be seen as the algebraic counterpart of the coherent category of towers of spaces from strong shape theory. Furthermore, it is involved in the description of the (Steenrod) homology group of a compact metrizable space as the homology group of a chain complex of abelian Polish groups.

1. Introduction

Given a category $\mathcal{C}$, the corresponding category of towers $\mathrm{tow}(\mathcal{C})$ is a full subcategory of the category $\mathrm{pro}(\mathcal{C})$ whose objects are towers in $\mathcal{C}$. A tower in $\mathcal{C}$ is simply an inverse sequence $(a^{(k)}, p^{(k,k')} : a^{(k')} \to a^{(k)})$ in $\mathcal{C}$, namely a contravariant functor from the ordered set ω of nonnegative integers to $\mathcal{C}$. A morphism in $\mathrm{tow}(\mathcal{C})$ from $(a^{(m)}, p^{(m,m')})$ to $(b^{(k)}, p^{(k,k')})$ is represented by a sequence $\left(m_k, f^{(k)}\right)$ where (m_k) is an increasing sequence in ω and $f^{(k)} : a^{(m_k)} \to b^{(k)}$ is a morphism in $\mathcal{C}$. Two such sequences $\left(m_k, f^{(k)}\right)$ and $\left(m'_k, f'^{(k)}\right)$ determine the same morphism if there exists an increasing sequence $(\tilde{m}_k)$ in ω such that $\tilde{m}_k \geq \max\{m_k, m'_k\}$ for $k \in \omega$ and $f^{(k)}p^{(m_k,\tilde{m}_k)} = f'^{(k)}p^{(m'_k,\tilde{m}_k)}$ for $k \in \omega$; see [1, Section 2.1] and [5, Section 1].

The category of towers arises naturally in the applications of category theory. In topology, it underpins the study of shape and (Steenrod) homology of compact metrizable spaces [1, 5]. In order to define the homology of a compact metrizable space X, one assigns to X a tower $\bar{C}_\bullet(X)$ of chain complexes. This tower is obtained by considering chain complexes associated with the nerves of an increasing sequence of finite open covers of X that is cofinal within the collection of all finite open covers of X ordered by refinement. In order for the corresponding map $X \mapsto \bar{C}_\bullet(X)$ to be a functor from the category of compact metrizable spaces to the category of towers $\mathrm{tow}(\mathcal{C})$, where $\mathcal{C}$ is the category of chain complexes of groups, one considers either the *simplicial chains of the Vietoris nerve* of the covers, as in [1, Section 8.2], or the *singular chains of the geometric realization* of the Čech nerve of the covers, as in [4, Section 19]. As remarked in [1, Section 8.2], "*We need Vietoris nerves rather than Čech nerves in order to obtain a functor* [...] *An interesting problem is the construction of a nerve that is "small" like the Čech nerve and "rigid" like the Vietoris nerve.*"

It is indeed useful in some contexts to consider the simplicial chains associated with the Čech nerve to obtain a "smaller" tower of chain complexes associated with a compact metrizable space. This is important for instance in [3] in order to realize the homology group of a compact metrizable space as the homology group chain complex of abelian Polish groups.

In this paper, we show that one can consider towers of chain complexes as objects of a category with a more generous notion of morphism than pro-morphism. We call such a category the coherent category of towers of chain complexes, and its arrows *coherent morphisms*. Roughly speaking, a coherent morphism is represented by a sequence $\left(m_k, f^{(k)}\right)$ as in the definition of a pro-morphism, where the requirement $f^{(k)}p^{(m_k,\tilde{m}_k)} = f'^{(k)}p^{(m'_k,\tilde{m}_k)}$ for $k \in \omega$ is replaced with the *coherence assumption* that $f^{(k)}p^{(m_k,\tilde{m}_k)}$ and $f'^{(k)}p^{(m'_k,\tilde{m}_k)}$ be chain homotopic, as witnessed by an explicitly given chain homotopy $f^{(k,k+1)}$. Two such sequences $\left(m_k, f^{(k)}, f^{(k,k+1)}\right)$ and $\left(m'_k, f'^{(k)}, f'^{(k,k+1)}\right)$ represent the same coherent morphism if they are connected by a 2-cell. It is proved in [3] that considering the chain complex associated with the *Čech nerve* of covers of a space one obtains a functor $X \mapsto C_\bullet(X)$ from the category of

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compact metrizable spaces to the coherent category of towers of chain complexes of *free finitely-generated* abelian groups.

We also consider *homotopy limits* of towers of chain complexes, and show that assigning to a tower $\boldsymbol{A}$ of chain complexes its homotopy limit holim$\boldsymbol{A}$ defines a functor from the coherent category of towers of chain complexes to the *homotopy category* of chain complexes.

The coherent category of the towers of chain complexes can be seen as an analogue of the coherent category of towers of topological spaces considered in [4, Section 1]. As we restrict ourselves to towers, we only need to impose "second order" coherence conditions, corresponding to 1-coherent mappings as in [4, Section 3]. The main results of this paper can be seen as algebraic versions of the results about the coherent category of towers of spaces from [4, Sections 1, 2, 3, and 18].

It seems likely that the results presented here can be formulated more generally within the context of strict ω-categories. However, in the proof we use more than the strict ω-category structure of chain complexes. We also need a "diagonal composition" of 2-cells yielding a 3-cell, which is not part of the definition of a strict ω-category.

2. The strict ω-category of chain complexes

We consider in this paper chain complexes (or, briefly, complexes) in an *additive category* $\mathcal{A}$; see [6, Section 5.5]. There is a natural structure of strict ω-category on the class of complexes; see [2, Chapter 1]. If A and B are complexes, then a 1-cell $A \to B$ is simply a chain map. If $f, f' : A \to B$ are chain maps, a 2-cell $L : f \Rightarrow f'$ is a chain homotopy. In this case, we also write

$$f \overset{L}{\Rightarrow} f'.$$

If $L, L' : f \Rightarrow f'$ is a chain homotopy, then a 3-cell $H : L \Rrightarrow L'$ is a sequence $(H_n)_{n\in\mathbb{Z}}$ of morphisms $H_n : A_n \to B_{n+2}$ such that $\partial H_n - H_{n-1}\partial = L'_n - L_n$, in which case we also write

$$L \overset{H}{\Rrightarrow} L'.$$

We denote by $-H$ the 3-cell $L' \Rrightarrow L'$ defined by $(-H)_n = -(H_n)$ for $n \in \mathbb{Z}$. More generally, for $k \geq 2$, we define a k-cell f with $s^k(f) = A$ and $t^k(f) = B$ and $(k-1)$-cells $s(f)$ and $t(f)$ as source and target to be a sequence of morphisms $f_n : A_n \to B_{n+k-1}$ for $n \in \mathbb{Z}$ such that

$$\partial f_n + (-1)^k f_{n-1}\partial = t(f)_n - s(f)_n .$$

Given a 0-cell A one defines its identity 1_A to be the 1-cell f defined by

$$f_n = 1_{A_n} : A_n \to A_n$$

for every $n \in \mathbb{Z}$. Similarly, for $k \geq 1$, given a k-cell g with $s^k(g) = A$ and $t^k(g) = B$ one defines its identity 1_g to be the $(k+1)$-cell f defined by

$$f_n = 0 : A_n \to B_{n+k-1}.$$

We now define composition of cells. If (g, f) is a pair of 1-cells composable along a 0-cell, one defines $gf = g \circ_0 f$ to be the 1-cell

$$(gf)_n = g_n \circ f_n$$

for $n \in \mathbb{Z}$. Suppose that (F, E) is a pair of 2-cells composable along 1-cells. Then define $F \circ_1 E := F + E$ where

$$(F+E)_n = F_n + E_n.$$

If (F, E) is a pair of 2-cells composable along 0-cells, then define $F \circ_0 E := Fr(E) + s(F)E$ where

$$(Fr(E) + s(F)E)_n = F_n t(E)_n + s(F)_{n+1} E_n$$

for $n \in \mathbb{Z}$. More generally, suppose that $0 \leq p < m$ and (F, E) is a pair of m-cells composable along p-cells. Define then

$$(F \circ_p E)_n = \begin{cases} F_n + E_n & \text{if } 0 < p < m, \\ F_n t(E)_n + s^m(F)_{n+m} E_n & \text{if } p = 0. \end{cases}$$

It is easy to verify that this indeed defines a strict ω-category.

If $L : f \Rightarrow f'$ is a chain homotopy between chain maps $f, f' : A \to B$, and $g : A' \to A$ and $h : B' \to B$, we define $hF = 1_h \circ_1 F$ and $Fg = F \circ_1 1_g$. Thus, by definition $hF : hf \Rightarrow hf'$ is given by $(hF)_n = h_{n+1}F_n$, and $Fg : fg \Rightarrow f'g$ is given by $(Fg)_n = F_n g_n$ for $n \in \mathbb{Z}$. One similarly define the composition of 3-cells with (identities of identities of) chain maps. Two chain maps $f, f' : A \to B$ are chain homotopic if there is a chain map $f \Rightarrow f'$. The *homotopy category* of complexes has complexes as objects and *chain homotopy equivalence classes* of 1-cells as arrows.

We also consider a different notion of composition between 2-cells, which we call *diagonal composition* along 0-cells.

Definition 2.1. Suppose that E and F are 2-cells with $s^2(F) = t^2(E)$. Then we define the *diagonal composition* FE to be the 3-cell

$$FE : Ft(E) \circ_1 s(F) E \Rrightarrow t(F) E \circ_1 Fs(E)$$

defined by

$$(FE)_n = F_{n+1} E_n$$

for $n \in \mathbb{Z}$.

3. The coherent category of towers of complexes

Suppose that $\mathcal{A}$ is an additive category. A tower of complexes in $\mathcal{A}$ is an inverse sequence $\boldsymbol{A} = \left(A^{(m)}, p^{(m,m+1)}\right)$ of complexes in $\mathcal{A}$ and chain maps $p^{(m,m+1)} : A^{(m+1)} \to A^{(m)}$. We then define $p^{(m_0,m_1)}$ to be the chain map $p^{(m_0,m_0+1)} \circ \cdots \circ p^{(m_1-1,m_1)}$ from $A^{(m_1)}$ to $A^{(m_0)}$, and $p^{(m_0,m_0)}$ to be the identity map of $A^{(m_0)}$. We now introduce the notion of 1-cell between towers of complexes. This can be seen as the analogue of 1-coherent mapping between towers of topological spaces as in [4, Sections 3].

Definition 3.1. Let $\boldsymbol{A} = \left(A^{(m)}, p^{(m,m+1)}\right)$ and $\boldsymbol{B} = \left(B^{(k)}, p^{(k,k+1)}\right)$ be towers of complexes. A 1-cell from $\boldsymbol{A}$ to $\boldsymbol{B}$ is given by a sequence $f = (m_k, f^{(k)}, f^{(k,k+1)})$ such that:

- (m_k) is an increasing sequence in ω,
- $f^{(k)} : A^{(m_k)} \to B^{(k)}$ is a chain map from $A^{(m_k)}$ to $B^{(k)}$ for every $k \in \omega$, and
- $f^{(k,k+1)} : p^{(k,k+1)} f^{(k+1)} \Rightarrow f^{(k)} p^{(m_k,m_{k+1})}$ is a chain homotopy.

One then defines for $k_0 \leq k_1$ the chain homotopy

$$f^{(k_0,k_1)} : p^{(k_0,k_1)} f^{(k_1)} \Rightarrow f^{(k_0)} p^{(m_{k_0},m_{k_1})}$$

by setting

$$f^{(k_0,k_0)} = 1_{f^{(k_0)}}$$

and, for $k_0 < k_1$,

$$f^{(k_0,k_1)} = \bigodot_{k=k_0}^{k_1-1} p^{(k_0,k)} f^{(k,k+1)} p^{(m_{k+1},m_{k_1})} = \sum_{k=k_0}^{k_1-1} p^{(k_0,k)} f^{(k,k+1)} p^{(m_{k+1},m_{k_1})}$$

where

$$\bigodot_{k=k_0}^{k_1-1} p^{(k_0,k)} f^{(k,k+1)} p^{(m_{k+1},m_{k_1})}$$

denotes the composition

$$f^{(k_0,k_0+1)} p^{(m_{k_0+1},m_{k_1})} \circ_1 p^{(k_0,k_0+1)} f^{(k_0+1,k_0+2)} p^{(m_{k_0+2},m_{k_1})} \circ_1 \cdots \circ_1 p^{(k_0,k_1-1)} f^{(k_1-1,k_1)}.$$

One can also define $f^{(k_0,k_1)}$ recursively by setting

$$f^{(k_0,k_1+1)} = f^{(k_0,k_1)} p^{(m_{k_1},m_{k_1+1})} \circ_1 p^{(k_0,k_1)} f^{(k_1,k_1+1)}.$$

The *identity* 1-cell of A is the 1-cell $\left(m_k, f^{(k)}, f^{(k,k+1)}\right)$ where $m_k = k$, $f^{(k)} = 1_{A^{(k)}}$, and $f^{(k,k+1)} = 1_{p^{(k,k+1)}}$.

We now define composition of 1-cells.

Definition 3.2. Suppose that $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ are towers of complexes, and $f = (m_k, f^{(k)}, f^{(k,k+1)})$ and $g = \left(k_t, g^{(t)}, h^{(t,t+1)}\right)$ are 1-cells from $\boldsymbol{A}$ to $\boldsymbol{B}$ and from $\boldsymbol{B}$ to $\boldsymbol{C}$, respectively. We define

$$gf = g \circ_0 f = \left(m_{k_t}, g^{(t)} f^{(k_t)}, g^{(t)} f^{(k_t,k_{t+1})} \circ_1 g^{(t,t+1)} f^{(k_{t+1})}\right).$$

Remark 3.3. Composition of 1-cells is *not* associative. However, we will show in Lemma 3.10 that is associative *up to* 2-cells.

The following notion can be seen as the analogue in this context of 1-coherent homotopies as in [4, Sections 3].

Definition 3.4. Let $\boldsymbol{A} = \left(A^{(m)}, p^{(m)}\right)$ and $\boldsymbol{B} = \left(B^{(k)}, p^{(k)}\right)$ be towers of complexes. Let $f = (m_k, f^{(k)}, f^{(k,k+1)})$ and $f' = \left(m'_k, f'^{(k)}, f'^{(k,k+1)}\right)$ be 1-cells from $\boldsymbol{A}$ to $\boldsymbol{B}$. A 2-cell from f to f' is a sequence $L = \left(\tilde{m}_k, L^{(k)}, L^{(k,k+1)}\right)$ such that:

- $(\tilde{m}_k)$ is an increasing sequence in ω such that $\tilde{m}_k \geq \max\{m_k, m'_k\}$ for $k \in \omega$,
- $L^{(k)} : f^{(k)}p^{(m_k,\tilde{m}_k)} \Rightarrow f'^{(k)}p^{(m'_k,\tilde{m}_k)}$ is a 2-cell,
- for every $k \in \omega$,

$$L^{(k,k+1)} : f'^{(k,k+1)}p^{(m'_{k+1},\tilde{m}_{k+1})} \circ_1 p^{(k,k+1)}L^{(k+1)} \Rrightarrow L^{(k)}p^{(\tilde{m}_k,\tilde{m}_{k+1})} \circ_1 f^{(k,k+1)}p^{(m_{k+1},\tilde{m}_{k+1})}.$$

One then defines for $k_0 < k_1$ the 3-cell

$$L^{(k_0,k_1)} : f'^{(k_0,k_1)}p^{(m'_{k_1},\tilde{m}_{k_1})} \circ_1 p^{(k_0,k_1)}L^{(k_1)} \Rrightarrow L^{(k_0)}p^{(\tilde{m}_{k_0},\tilde{m}_{k_1})} \circ_1 f^{(k_0,k_1)}p^{(m_{k_1},\tilde{m}_{k_1})}$$

recursively by setting $L^{(k_0,k_0)}$ to be equal to $1_{L^{(k_0)}}$, and $L^{(k_0,k_1+1)}$ to be equal to $H'^{(k_0,k_1+1)} \circ_2 H^{(k_0,k_1+1)}$ where

$$H'^{(k_0,k_1+1)} = L^{(k_0,k_1)}p^{(\tilde{m}_{k_1},\tilde{m}_{k_1+1})} \circ_1 1_{p^{(k_0,k_1)}f^{(k_1,k_1+1)}p^{(m_{k_1},\tilde{m}_{k_1+1})}}$$

and

$$H^{(k_0,k_1+1)} = 1_{f'^{(k_0,k_1)}p^{(m'_{k_1},\tilde{m}_{k_1+1})}} \circ_1 p^{(k_0,k_1)}L^{(k_1,k_1+1)}.$$

Remark 3.5. Notice that

$$p^{(k_0,k_1)}f^{(k_1)}p^{(m_{k_1},\tilde{m}_{k_1})} \overset{f^{(k_0,k_1)}p^{(m_{k_1},\tilde{m}_{k_1})}}{\Rightarrow} f^{(k_0)}p^{(m_{k_0},\tilde{m}_{k_1})} \overset{L^{(k_0)}p^{(\tilde{m}_{k_0},\tilde{m}_{k_1})}}{\Rightarrow} f'^{(k_1)}p^{(m'_{k_1},\tilde{m}_{k_1})}$$

and

$$p^{(k_0,k_1)}f^{(k_1)}p^{(m_{k_1},\tilde{m}_{k_1})} \overset{p^{(k_0,k_1)}L^{(k_1)}}{\Rightarrow} p^{(k_0,k_1)}f'^{(k_1)}p^{(m_{k'_1},\tilde{m}_{k_1})} \overset{f'^{(k_0,k_1)}p^{(m'_{k_1},\tilde{m}_{k_1})}}{\Rightarrow} f'^{(k_1)}p^{(m'_{k_1},\tilde{m}_{k_1})}.$$

Therefore

$$f'^{(k_0,k_1)}p^{(m'_{k_1},\tilde{m}_{k_1})} \circ_1 p^{(k_0,k_1)}L^{(k_1)}$$

and

$$L^{(k_0)}p^{(\tilde{m}_{k_0},\tilde{m}_{k_1})} \circ_1 f^{(k_0,k_1)}p^{(m_{k_1},\tilde{m}_{k_1})}$$

are 2-cells from $p^{(k_0,k_1)}f^{(k_1)}p^{(m_{k_1},\tilde{m}_{k_1})}$ to $f'^{(k_1)}p^{(m'_{k_1},\tilde{m}_{k_1})}$.

One can prove by induction that $L^{(k_0,k_1)}$ is a 3-cell

$$L^{(k_0,k_1)} : f'^{(k_0,k_1)}p^{(m'_{k_1},\tilde{m}_{k_1})} \circ_1 p^{(k_0,k_1)}L^{(k_1)} \Rrightarrow L^{(k_0)}p^{(\tilde{m}_{k_0},\tilde{m}_{k_1})} \circ_1 f^{(k_0,k_1)}p^{(m_{k_1},\tilde{m}_{k_1})}.$$

Indeed, suppose that this is the case for some $k_0 < k_1$. By definition, we also have that

$$L^{(k_1,k_1+1)} : f'^{(k_1,k_1+1)}p^{(m'_{k_1+1},\tilde{m}_{k_1+1})} \circ_1 p^{(k_1,k_1+1)}L^{(k_1+1)} \Rrightarrow L^{(k_1)}p^{(\tilde{m}_{k_1},\tilde{m}_{k_1+1})} \circ_1 f^{(k_1,k_1+1)}p^{(m_{k_1+1},\tilde{m}_{k_1+1})}$$

Hence, letting $H^{(k_0,k_1+1)}$ and $H'^{(k_0,k_1+1)}$ be as above,

$$\begin{aligned}
& f'^{(k_0,k_1+1)}p^{(m'_{k_1+1},\tilde{m}_{k_1+1})} \circ_1 p^{(k_0,k_1+1)}L^{(k_1+1)} \\
= \; & \left(f'^{(k_0,k_1)}p^{(m'_{k_1},m'_{k_1+1})} \circ_1 p^{(k_0,k_1)}f'^{(k_1,k_1+1)}\right)p^{(m'_{k_1+1},\tilde{m}_{k_1+1})} \circ_1 p^{(k_0,k_1+1)}L^{(k_1+1)} \\
= \; & f'^{(k_0,k_1)}p^{(m'_{k_1},\tilde{m}_{k_1+1})} \circ_1 p^{(k_0,k_1)}f'^{(k_1,k_1+1)}p^{(m'_{k_1+1},\tilde{m}_{k_1+1})} \circ_1 p^{(k_0,k_1+1)}L^{(k_1+1)} \\
= \; & f'^{(k_0,k_1)}p^{(m'_{k_1},\tilde{m}_{k_1+1})} \circ_1 p^{(k_0,k_1)}\left(f'^{(k_1,k_1+1)}p^{(m'_{k_1+1},\tilde{m}_{k_1+1})} \circ_1 p^{(k_1,k_1+1)}L^{(k_1+1)}\right) \\
\overset{H^{(k_0,k_1+1)}}{\Rrightarrow} \; & f'^{(k_0,k_1)}p^{(m'_{k_1},\tilde{m}_{k_1+1})} \circ_1 p^{(k_0,k_1)}\left(L^{(k_1)}p^{(\tilde{m}_{k_1},\tilde{m}_{k_1+1})} \circ_1 f^{(k_1,k_1+1)}p^{(m_{k_1},\tilde{m}_{k_1+1})}\right) \\
= \; & f'^{(k_0,k_1)}p^{(m'_{k_1},\tilde{m}_{k_1+1})} \circ_1 p^{(k_0,k_1)}L^{(k_1)}p^{(\tilde{m}_{k_1},\tilde{m}_{k_1+1})} \circ_1 p^{(k_0,k_1)}f^{(k_1,k_1+1)}p^{(m_{k_1},\tilde{m}_{k_1+1})} \\
= \; & \left(f'^{(k_0,k_1)}p^{(m'_{k_1},\tilde{m}_{k_1})} \circ_1 p^{(k_0,k_1)}L^{(k_1)}\right)p^{(\tilde{m}_{k_1},\tilde{m}_{k_1+1})} \circ_1 p^{(k_0,k_1)}f^{(k_1,k_1+1)}p^{(m_{k_1},\tilde{m}_{k_1+1})} \\
\overset{H'^{(k_0,k_1+1)}}{\Rrightarrow} \; & \left(L^{(k_0)}p^{(\tilde{m}_{k_0},\tilde{m}_{k_1})} \circ_1 f^{(k_0,k_1)}p^{(m_{k_1},\tilde{m}_{k_1})}\right)p^{(\tilde{m}_{k_1},\tilde{m}_{k_1+1})} \circ_1 p^{(k_0,k_1)}f^{(k_1,k_1+1)}p^{(m_{k_1},\tilde{m}_{k_1+1})} \\
= \; & L^{(k_0)}p^{(\tilde{m}_{k_0},\tilde{m}_{k_1+1})} \circ_1 f^{(k_0,k_1)}p^{(m_{k_1},\tilde{m}_{k_1+1})} \circ_1 p^{(k_0,k_1)}f^{(k_1,k_1+1)}p^{(m_{k_1},\tilde{m}_{k_1+1})} \\
= \; & L^{(k_0)}p^{(\tilde{m}_{k_0},\tilde{m}_{k_1+1})} \circ_1 \left(f^{(k_0,k_1)}p^{(m_{k_1},m_{k_1+1})} \circ_1 p^{(k_0,k_1)}f^{(k_1,k_1+1)}\right)p^{(m_{k_1},\tilde{m}_{k_1+1})} \\
= \; & L^{(k_0)}p^{(\tilde{m}_{k_0},\tilde{m}_{k_1+1})} \circ_1 f^{(k_0,k_1+1)}p^{(m_{k_1},\tilde{m}_{k_1+1})}
\end{aligned}$$

This concludes the proof.

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two towers of complexes in $\mathcal{A}$. We say that two 1-cells $f, f' : \boldsymbol{A} \to \boldsymbol{B}$ *represent the same coherent morphism* if there exists a 2-cell $L : f \Rightarrow f'$. By Lemma 3.9 below, this defines an equivalence relation on the set of 1-cells from $\boldsymbol{A}$ to $\boldsymbol{B}$. Given a 1-cell f we let $[f]$ be its equivalence class, and we call $[f]$ the coherent morphism from $\boldsymbol{A}$ to $\boldsymbol{B}$ represented by f. The composition of the coherent morphism $[f]$ from $\boldsymbol{A}$ to $\boldsymbol{B}$ with the coherent morphism $[g]$ from $\boldsymbol{B}$ to $\boldsymbol{C}$ is by definition $[g \circ_0 f]$. The identity coherent morphism of $\boldsymbol{A}$ is $[1_{\boldsymbol{A}}]$, where $1_{\boldsymbol{A}}$ is the identity 1-cell of $\boldsymbol{A}$. The rest of this section is dedicated to the proof of the following result.

Theorem 3.6. *Let $\mathcal{A}$ be an additive category. The composition operation between coherent morphisms is well-defined, and yields a category with complexes in $\mathcal{A}$ as objects and coherent morphisms as arrows.*

For future use, we introducing some terminology concerning special kinds of 1-cells.

Definition 3.7. Let $\boldsymbol{A}$ be a tower of complexes, and (m_k) be an increasing sequence in ω. Define $\boldsymbol{B}$ to be the tower given by $B^{(k)} = A^{(m_k)}$. The *cofinal 1-cell* $\pi(m_k) : \boldsymbol{A} \to \boldsymbol{B}$ is the 1-cell $(m_k, 1_{A^{(m_k)}}, 1_{p^{(m_k,m_{k+1})}})$.

Definition 3.8. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be towers of complexes. A *level* 1-cell $\boldsymbol{A} \to \boldsymbol{B}$ is a 1-cell $f = (m_k, f^{(k)}, f^{(k,k+1)}) : \boldsymbol{A} \to \boldsymbol{B}$ such that $m_k = k$ for every $k \in \omega$.

We begin with showing that the relation of being connected by a 2-cell is an equivalence relation.

Lemma 3.9. Suppose that $\boldsymbol{A}$, $\boldsymbol{B}$ are towers of complexes, $f = (m_k, f^{(k)}, f^{(k,k+1)})$, $f' = (m'_k, f'^{(k)}, f'^{(k,k+1)})$ and $f'' = (m''_k, f''^{(k)}, f''^{(k,k+1)})$ are 1-cells from $\boldsymbol{A}$ to $\boldsymbol{B}$. If there exist 2-cells $f \Rightarrow f'$ and $f' \Rightarrow f''$, then there exists a 2-cell $f \Rightarrow f''$.

Proof. Suppose that $L = \left(\tilde{m}_k, L^{(k)}, L^{(k,k+1)}\right)$, and $L' = \left(\tilde{m}'_k, L'^{(k)}, L'^{(k,k+1)}\right)$ are 2-cells from f to f' and from f' to f'', respectively. We define

$$L' \circ_1 L = \left(\ell_k, S^{(k)}, S^{(k,k+1)}\right)$$

where $\ell_k = \max\{\tilde{m}_k, \tilde{m}'_k\}$, $S^{(k)}$ is the 2-cell

$$L'^{(k)}p^{\left(\tilde{m}'_k,\ell_k\right)} \circ_1 L^{(k)}p^{(\tilde{m}_k,\ell_k)}$$

and $S^{(k,k+1)}$ is the 3-cell $H'^{(k,k+1)} \circ_2 H^{(k,k+1)}$, where

$$H^{(k,k+1)} = L'^{(k,k+1)}p^{(\tilde{m}'_{k+1},\ell_{k+1})} \circ_1 1_{p^{(k,k+1)}L^{(k+1)}p^{(\tilde{m}_{k+1},\ell_{k+1})}}$$

and

$$H'^{(k,k+1)} = 1_{L'^{(k)}p^{(\tilde{m}'_k,\ell_{k+1})}} \circ_1 L^{(k,k+1)}p^{(\tilde{m}_{k+1},\ell_{k+1})}.$$

We claim that $L' \circ_1 L : f \Rightarrow f''$ is a 2-cell.

By assumption, we have that $L : f \Rightarrow f'$ and $L' : f' \Rightarrow f''$ are 2-cells. Hence, we have that

$$L^{(k)} : f^{(k)}p^{(m_k,\tilde{m}_k)} \Rightarrow f'^{(k)}p^{(m'_k,\tilde{m}_k)}$$

$$L'^{(k)} : f'^{(k)}p^{(m'_k,\tilde{m}'_k)} \Rightarrow f''^{(k)}p^{(m''_k,\tilde{m}'_k)}$$

are chain homotopies, and

$$L^{(k,k+1)} : f'^{(k,k+1)}p^{(m'_{k+1},\tilde{m}_{k+1})} \circ_1 p^{(k,k+1)}L^{(k+1)} \Rrightarrow L^{(k)}p^{(\tilde{m}_k,\tilde{m}_{k+1})} \circ_1 f^{(k,k+1)}p^{(m_{k+1},\tilde{m}_{k+1})}$$

and

$$L'^{(k,k+1)} : f''^{(k,k+1)}p^{(m''_{k+1},\tilde{m}'_{k+1})} \circ_1 p^{(k,k+1)}L'^{(k+1)} \Rrightarrow L'^{(k)}p^{(\tilde{m}'_k,\tilde{m}'_{k+1})} \circ_1 f'^{(k,k+1)}p^{(m'_{k+1},\tilde{m}'_{k+1})}$$

are 3-cells. Hence, we have

$$\begin{array}{cl} & f^{(k)}p^{(m_k,\ell_k)} \\ = & f^{(k)}p^{(m_k,\tilde{m}_k)}p^{(\tilde{m}_k,\ell_k)} \\ \overset{L^{(k)}p^{(\tilde{m}_k,\ell_k)}}{\Rightarrow} & f'^{(k)}p^{(m'_k,\ell_k)} \\ = & f'^{(k)}p^{(m'_k,\tilde{m}'_k)}p^{(\tilde{m}'_k,\ell_k)} \\ \overset{L'^{(k)}p^{(\tilde{m}'_k,\ell_k)}}{\Rightarrow} & f''^{(k)}p^{(m''_k,\ell_k)}. \end{array}$$

Thus,

$$S^{(k)} = L'^{(k)}p^{\left(\tilde{m}'_k,\ell_k\right)} \circ_1 L^{(k)}p^{(\tilde{m}_k,\ell_k)} : f^{(k)}p^{(m_k,\ell_k)} \Rightarrow f''^{(k)}p^{(m''_k,\ell_k)}$$

is a chain homotopy.

Similarly, we have that

$$\begin{array}{cl} & f''^{(k,k+1)}p^{(m''_{k+1},\ell_{k+1})} \circ_1 p^{(k,k+1)}S^{(k+1)} \\ = & f''^{(k,k+1)}p^{(m''_{k+1},\ell_{k+1})} \circ_1 p^{(k,k+1)}L'^{(k+1)}p^{\left(\tilde{m}'_{k+1},\ell_{k+1}\right)} \circ_1 p^{(k,k+1)}L^{(k+1)}p^{(\tilde{m}_{k+1},\ell_{k+1})} \\ = & \left(f''^{(k,k+1)}p^{(m''_{k+1},\tilde{m}'_{k+1})} \circ_1 p^{(k,k+1)}L'^{(k+1)}\right)p^{(\tilde{m}'_{k+1},\ell_{k+1})} \circ_1 p^{(k,k+1)}L^{(k+1)}p^{(\tilde{m}_{k+1},\ell_{k+1})} \\ \overset{H^{(k,k+1)}}{\Rrightarrow} & \left(L'^{(k)}p^{(\tilde{m}'_k,\tilde{m}'_{k+1})} \circ_1 f'^{(k,k+1)}p^{(m'_{k+1},\tilde{m}'_{k+1})}\right)p^{(\tilde{m}'_{k+1},\ell_{k+1})} \circ_1 p^{(k,k+1)}L^{(k+1)}p^{(\tilde{m}_{k+1},\ell_{k+1})} \\ = & L'^{(k)}p^{(\tilde{m}'_k,\ell_{k+1})} \circ_1 f'^{(k,k+1)}p^{(m'_{k+1},\ell_{k+1})} \circ_1 p^{(k,k+1)}L^{(k+1)}p^{(\tilde{m}_{k+1},\ell_{k+1})} \\ = & L'^{(k)}p^{(\tilde{m}'_k,\ell_{k+1})} \circ_1 \left(f'^{(k,k+1)}p^{(m'_{k+1},\tilde{m}_{k+1})} \circ_1 p^{(k,k+1)}L^{(k+1)}\right)p^{(\tilde{m}_{k+1},\ell_{k+1})} \\ \overset{H'^{(k,k+1)}}{\Rrightarrow} & L'^{(k)}p^{(\tilde{m}'_k,\ell_{k+1})} \circ_1 \left(L^{(k)}p^{(\tilde{m}_k,\tilde{m}_{k+1})} \circ_1 f^{(k,k+1)}p^{(m_{k+1},\tilde{m}_{k+1})}\right)p^{(\tilde{m}_{k+1},\ell_{k+1})} \\ = & L'^{(k)}p^{(\tilde{m}'_k,\ell_{k+1})} \circ_1 L^{(k)}p^{(\tilde{m}_k,\ell_{k+1})} \circ_1 f^{(k,k+1)}p^{(m_{k+1},\ell_{k+1})} \\ = & S^{(k)}p^{(\ell_k,\ell_{k+1})} \circ_1 f^{(k,k+1)}p^{(m_{k+1},\ell_{k+1})}. \end{array}$$

This shows that

$$S^{(k,k+1)} = H^{(k,k+1)} \circ_1 H'^{(k,k+1)}$$

is a 3-cell

$$f''^{(k,k+1)}p^{(m''_{k+1},\ell_{k+1})} \circ_1 p^{(k,k+1)}S^{(k+1)} \Rrightarrow S^{(k)}p^{(\ell_k,\ell_{k+1})} \circ_1 f^{(k,k+1)}p^{(m_{k+1},\ell_{k+1})}.$$

This concludes the proof. □

We now verify that composition of 1-cells is associative *up to* 2-cells.

Lemma 3.10. Suppose that $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ are towers of complexes, and $f = (m_k, f^{(k)}, f^{(k,k+1)})$ and $g = \left(k_t, g^{(t)}, g^{(t,t+1)}\right)$ are 1-cells from $\boldsymbol{A}$ to $\boldsymbol{B}$ and from $\boldsymbol{B}$ to $\boldsymbol{C}$, respectively. For $t_0 < t_1$ define

$$w^{(t_0,t_1)} = \bigodot_{t=t_0}^{t_1-1} \left(p^{(t_0,t)} g^{(t)} f^{(k_t,k_{t+1})} p^{(m_{k_{t+1}}, m_{k_{t_1}})} \circ_1 p^{(t_0,t)} g^{(t,t+1)} f^{(k_{t+1})} p^{(m_{k_{t+1}}, m_{k_{t_1}})} \right)$$

and

$$w'^{(t_0,t_1)} = g^{(t_0)} f^{(k_{t_0},k_{t_1})} \circ_1 g^{(t_0,t_1)} f^{(k_{t_1})}.$$

Then there is a 3-cell $L^{(t_0,t_1)} : w^{(t_0,t_1)} \Rrightarrow w'^{(t_0,t_1)}$.

Proof. Notice that $w^{(t_0,t_1)}$ and $w'^{(t_0,t_1)}$ are 2-cells

$$p^{(t_0,t_1)} g^{(t_1)} f^{(k_{t_1})} \Rightarrow g^{(t_0)} f^{(k_{t_0})} p^{(m_{k_{t_0}}, m_{k_{t_1}})}.$$

Recall the definition of *diagonal composition* of 2-cells as in Definition 2.1. Define the 3-cell

$$L^{(t_0,t_0+1)} = 1_{w^{(t_0,t_0+1)}}.$$

Define also the 3-cell $L^{(t_0,t_0+2)}$ to be equal to

$$1_{g^{(t_0)} f^{(k_{t_0},k_{t_0+1})} p^{(m_{k_{t_0+1}}, m_{k_{t_0+2}})}} \circ_1 g^{(t_0,t_0+1)} f^{(k_{t_0+1},k_{t_0+2})} \circ_1 1_{p^{(t_0,t_0+1)} g^{(t_0+1,t_0+2)} f^{(k_{t_0+2})}}$$

and, recursively, for $t_0 + 2 < t_1$, the 3-cell $L^{(t_0,t_1+1)}$ to be equal to

$$H'^{(t_0,t_1+1)} \circ_2 H^{(t_0,t_1+1)}$$

where

$$H'^{(t_0,t_1+1)} = 1_{g^{(t_0)} f^{(k_{t_0},k_{t_1})} p^{(m_{k_{t_1}}, m_{k_{t_1+1}})}} \circ_1 g^{(t_0,t_0+1)} f^{(k_{t_0+1},k_{t_1+1})} \circ_1 1_{p^{(t_0,t_1)} g^{(t_1,t_1+1)} f^{(k_{t_1+1})}}$$

and

$$H^{(t_0,t_1+1)} = L^{(t_0,t_1)} p^{(m_{k_{t_1}}, m_{k_{t_1+1}})} \circ_1 1_{p^{(t_0,t_1)} w^{(t_1,t_1+1)}}.$$

We claim that $L^{(t_0,t_1)}$ is a 3-cell from $w^{(t_0,t_1)}$ to $w'^{(t_0,t_1)}$ for every $t_0 < t_1$.

Suppose initially that $t_1 = t_0 + 1$. In this case, we have

$$\begin{aligned} w^{(t_0,t_0+1)} &= g^{(t_0)} f^{(k_{t_0},k_{t_0+1})} \circ_1 p^{(t_0,t_0)} g^{(t_0,t_0+1)} f^{(k_{t_0+1})} \\ &= g^{(t_0)} f^{(k_{t_0},k_{t_1})} \circ_1 g^{(t_0,t_1)} f^{(k_{t_1})} = w'^{(t_0,t_1)}. \end{aligned}$$

Thus, $L^{(t_0,t_0+1)} = 1_{w^{(t_0,t_0+1)}}$ is a 3-cell from $w^{(t_0,t_0+1)}$ to $w'^{(t_0,t_0+1)}$. Suppose now that $t_1 = t_0 + 2$. In this case, by definition we have that $L^{(t_0,t_0+2)}$ is equal to

$$1_{g^{(t_0)} f^{(k_{t_0},k_{t_0+1})} p^{(m_{k_{t_0+1}}, m_{k_{t_0+2}})}} \circ_2 g^{(t_0,t_0+1)} f^{(k_{t_0+1},k_{t_0+2})} \circ_2 1_{p^{(t_0,t_0+1)} g^{(t_0+1,t_0+2)} f^{(k_{t_0+2})}}.$$

Furthermore,

$$
\begin{aligned}
& w^{(t_0,t_0+2)} \\
= \quad & \left(g^{(t_0)} f^{(k_{t_0},k_{t_0+1})} \circ_1 g^{(t_0,t_0+1)} f^{(k_{t_0+1})}\right) p^{(m_{k_{t_0+1}},m_{k_{t_0+2}})} \\
& \circ_1 p^{(t_0,t_0+1)} \left(g^{(t_0+1)} f^{(k_{t_0+1},k_{t_0+2})} \circ_1 g^{(t_0+1,t_0+2)} f^{(k_{t_0+2})}\right) \\
= \quad & \left(g^{(t_0)} f^{(k_{t_0},k_{t_0+1})} p^{(m_{k_{t_0+1}},m_{k_{t_0+2}})} \circ_1 g^{(t_0,t_0+1)} f^{(k_{t_0+1})} p^{(m_{k_{t_0+1}},m_{k+2})}\right) \\
& \circ_1 \left(p^{(t_0,t_0+1)} g^{(t_0+1)} f^{(k_{t_0+1},k_{t_0+2})} \circ_1 p^{(t_0,t_0+1)} g^{(t_0+1,t_0+2)} f^{(k_{t_0+2})}\right) \\
= \quad & \left(g^{(t_0)} f^{(k_{t_0},k_{t_0+1})} \circ_1 g^{(t_0,t_0+1)} f^{(k_{t_0+1})}\right) p^{(m_{k_{t_0+1}},m_{k_{t_0+2}})} \\
& \circ_1 p^{(t_0,t_0+1)} \left(g^{(t_0+1)} f^{(k_{t_0+1},k_{t_0+2})} \circ_1 g^{(t_0+1,t_0+2)} f^{(k_{t_0+2})}\right) \\
= \quad & g^{(t_0)} f^{(k_{t_0},k_{t_0+1})} p^{(m_{k_{t_0+1}},m_{k_{t_0+2}})} \circ_1 g^{(t_0,t_0+1)} f^{(k_{t_0+1})} p^{(m_{k_{t_0+1}},m_{k_{t_0+2}})} \\
& \circ_1 p^{(t_0,t_0+1)} g^{(t_0+1)} f^{(k_{t_0+1},k_{t_0+2})} \circ p^{(t_0,t_0+1)} g^{(t_0+1,t_0+2)} f^{(k_{t_0+2})} \\
= \quad & g^{(t_0)} f^{(k_{t_0},k_{t_0+1})} p^{(m_{k_{t_0+1}},m_{k_{t_0+2}})} \circ_1 g^{(t_0,t_0+1)} t\left(f^{(k_{t_0+1},k_{t_0+2})}\right) \\
& \circ_1 s\left(g^{(t_0,t_0+1)}\right) f^{(k_{t_0+1},k_{t_0+2})} \circ p^{(t_0,t_0+1)} g^{(t_0+1,t_0+2)} f^{(k_{t_0+2})} \\
\overset{L^{(t_0,t_0+2)}}{\Rrightarrow} \quad & g^{(t_0)} f^{(k_{t_0},k_{t_0+1})} p^{(m_{k_{t_0+1}},m_{k_{t_0+2}})} \circ_1 t\left(g^{(t_0,t_0+1)}\right) f^{(k_{t_0+1},k_{t_0+2})} \\
& \circ_1 g^{(t_0,t_0+1)} s\left(f^{(k_{t_0+1},k_{t_0+2})}\right) \circ p^{(t_0,t_0+1)} g^{(t_0+1,t_0+2)} f^{(k_{t_0+2})} \\
= \quad & g^{(t_0)} f^{(k_{t_0},k_{t_0+1})} p^{(m_{k_{t_0+1}},m_{k_{t_0+2}})} \circ_1 g^{(t_0)} p^{(k_{t_0},k_{t_0+1})} f^{(k_{t_0+1},k_{t_0+2})} \\
& \circ_1 g^{(t_0,t_0+1)} p^{(k_{t_0+1},k_{t_0+2})} f^{(k_{t_0+2})} \circ_1 p^{(t_0,t_0+1)} g^{(t_0+1,t_0+2)} f^{(k_{t_0+2})} \\
= \quad & g^{(t_0)} f^{(k_{t_0},k_{t_0+2})} \circ_1 g^{(t_0,t_2)} f^{(k_{t_0+2})} \\
= \quad & w'^{(t_0,t_0+2)}.
\end{aligned}
$$

This concludes the proof in the case when $t_1 = t_0 + 2$.

We now prove that, if the conclusion holds for t_0, t_1, then it holds for $t_0, t_1 + 1$. We have that

$$
\begin{aligned}
& w^{(t_0,t_1+1)} \\
= \quad & \bigodot_{t=t_0}^{t_1} \left(p^{(t_0,t)} g^{(t)} f^{(k_t,k_{t+1})} p^{(m_{k_{t+1}},m_{k_{t_1+1}})} \circ_1 p^{(t_0,t)} g^{(t,t+1)} f^{(k_{t+1})} p^{(m_{k_{t+1}},m_{k_{t_1}+1})}\right) \\
= \quad & \bigodot_{t=t_0}^{t_1-1} \left(p^{(t_0,t)} g^{(t)} f^{(k_t,k_{t+1})} p^{(m_{k_{t+1}},m_{k_{t_1}})} \circ_1 p^{(t_0,t)} g^{(t,t+1)} f^{(k_{t+1})} p^{(m_{k_{t+1}},m_{k_{t_1}})}\right) p^{(m_{k_{t_1}},m_{k_{t_1+1}})} \\
& \circ_1 p^{(t_0,t_1)} \left(g^{(t_1)} f^{(k_{t_1},k_{t_1+1})} \circ_1 g^{(t_1,t_1+1)} f^{(k_{t_1+1})}\right) \\
= \quad & w^{(t_0,t_1)} p^{(m_{k_{t_1}},m_{k_{t_1+1}})} \circ_1 p^{(t_0,t_1)} w^{(t_1,t_1+1)}.
\end{aligned}
$$

By definition, we have that $L^{(t_0,t_1+1)}$ is equal to

$$H'^{(t_0,t_1+1)} \circ_2 H^{(t_0,t_1+1)}$$

where

$$H'^{(t_0,t_1+1)} = 1_{g^{(t_0)} f^{(k_{t_0},k_1)} p^{(m_{k_{t_1}},m_{k_{t_1+1}})}} \circ_1 g^{(t_0,t_0+1)} f^{(k_{t_0+1},k_{t_1+1})} \circ_1 1_{p^{(t_0,t_1)} g^{(t_1,t_1+1)} f^{(k_{t_1+1})}}$$

and

$$H^{(t_0,t_1+1)} = L^{(t_0,t_1)} p^{(m_{k_{t_1}},m_{k_{t_1+1}})} \circ_1 1_{p^{(t_0,t_1)} w^{(t_1,t_1+1)}}.$$

By the inductive hypothesis, we have that

$$\begin{aligned}
& w^{(t_0,t_1+1)} \\
= \quad & w^{(t_0,t_1)} p^{(m_{k_{t_1}},m_{k_{t_1+1}})} \circ_1 p^{(t_0,t_1)} w^{(t_1,t_1+1)} \\
\overset{H^{(t_0,t_1+1)}}{\Rrightarrow} \quad & g^{(t_0)} f^{(k_{t_0},k_{t_1})} p^{(m_{k_{t_1}},m_{k_{t_1+1}})} \circ_1 g^{(t_0,t_0+1)} f^{(k_{t_0+1})} p^{(m_{k_{t_1}},m_{k_{t_1+1}})} \circ_1 p^{(t_0,t_1)} g^{(t_1)} f^{(k_{t_1},k_{t_1+1})} \circ_1 p^{(t_0,t_1)} g^{(t_1,t_1+1)} f^{(k_{t_1+1})} \\
= \quad & g^{(t_0)} f^{(k_{t_0},k_{t_1})} p^{(m_{k_{t_1}},m_{k_{t_1+1}})} \circ_1 g^{(t_0,t_0+1)} t\left(f^{(k_{t_0+1},k_{t_1+1})}\right) \circ_1 s\left(g^{(t_0,t_1)}\right) f^{(k_{t_1},k_{t_1+1})} \circ_1 p^{(t_0,t_1)} g^{(t_1,t_1+1)} f^{(k_{t_1+1})} \\
\overset{H'^{(t_0,t_1+1)}}{\Rrightarrow} \quad & g^{(t_0)} f^{(k_{t_0},k_{t_1})} p^{(m_{k_{t_1}},m_{k_{t_1+1}})} \circ_1 t\left(g^{(t_0,t_0+1)}\right) f^{(k_{t_0+1},k_{t_1+1})} \circ_1 g^{(t_0,t_1)} s\left(f^{(k_{t_1},k_{t_1+1})}\right) \circ_1 p^{(t_0,t_1)} g^{(t_1,t_1+1)} f^{(k_{t_1+1})} \\
= \quad & g^{(t_0)} f^{(k_{t_0},k_{t_1})} p^{(m_{k_{t_1}},m_{k_{t_1+1}})} \circ_1 g^{(t_0)} p^{(t_0,t_0+1)} f^{(k_{t_0+1},k_{t_1+1})} \circ_1 g^{(t_0,t_1)} p^{(k_{t_1},k_{t_1+1})} f^{(k_{t_1+1})} \circ_1 p^{(t_0,t_1)} g^{(t_1,t_1+1)} f^{(k_{t_1+1})} \\
= \quad & g^{(t_0)} f^{(k_{t_0},k_{t_1+1})} \circ_1 g^{(t_0,t_1+1)} f^{(k_{t_1+1})} \\
= \quad & w'^{(t_0,t_1+1)}.
\end{aligned}$$

Thus $L^{(t_0,t_1+1)} = H^{(t_0,t_1+1)} \circ_2 H'^{(t_0,t_1+1)}$ is a 3-cell $w^{(t_0,t_1+1)} \Rrightarrow w'^{(t_0,t_1+1)}$. This concludes the proof. □

The following lemma can be seen as the analogue in our context of the result [4, Theorem 2.8] about associativity of composition of coherent mappings *up to coherent homotopies*.

Lemma 3.11. Suppose that $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D}$ are towers of complexes, and $f = (m_k, f^{(k)}, f^{(k,k+1)})$, $g = \left(k_t, g^{(t)}, g^{(t,t+1)}\right)$, and $h = \left(t_\ell, h^{(\ell)}, h^{(\ell,\ell+1)}\right)$ are 1-cells from $\boldsymbol{A}$ to $\boldsymbol{B}$, from $\boldsymbol{B}$ to $\boldsymbol{C}$, and from $\boldsymbol{C}$ to $\boldsymbol{D}$, respectively. Then there exists a 2-cell $(hg)\, f \Rightarrow h\,(gf)$.

Proof. It is clear that the composition of *level* 1-cells is associative. Therefore, it suffices to consider the case when h is the cofinal 1-cell associated to the sequence (t_ℓ). In this case, $C^{(\ell)} = B^{(t_\ell)}$, $h^{(\ell)} = 1_{C^{(\ell)}}$ and $h^{(\ell,\ell+1)} = 0$.

We define $\alpha = \left(m_{k_t}, \alpha^{(t)}, \alpha^{(t,t+1)}\right)$ to be the composition

$$gf = \left(m_{k_t}, g^{(t)} f^{(k_t)}, g^{(t)} f^{(k_t,k_{t+1})} \circ_1 g^{(t,t+1)} f^{(k_{t+1})}\right)$$

Let also $\beta = \left(k_{t_\ell}, \beta^{(\ell)}, \beta^{(\ell,\ell+1)}\right)$ be the composition

$$hg = (k_{t_\ell}, g^{(t_\ell)}, g^{(t_\ell,t_{\ell+1})}).$$

We have that

$$h\,(gf) = \left(t_\ell, h^{(\ell)}, h^{(\ell,\ell+1)}\right) \circ (m_{k_t}, \alpha^{(t)}, \alpha^{(t,t+1)}) = \left(m_{k_{t_\ell}}, \alpha^{(t_\ell)}, \alpha^{(t_\ell,t_{\ell+1})}\right)$$

where

$$\alpha^{(t_\ell)} = g^{(t_\ell)} f^{(k_{t_\ell})}$$

and

$$\alpha^{(t_\ell,t_{\ell+1})} = \sum_{t=t_\ell}^{t_{\ell+1}-1} p^{(t_\ell,t)} \left(g_{n+1}^{(t)} f_n^{(k_t,k_{t+1})} \circ_1 g_n^{(t,t+1)} f_n^{(k_{t+1})}\right) p^{(m_{k_{t+1}},m_{k_{t_{\ell+1}}})}.$$

We also have that

$$\begin{aligned}
(hg)\, f &= \left(k_{t_\ell}, \beta^{(\ell)}, \beta^{(\ell,\ell+1)}\right) \circ (m_k, f^{(k)}, f^{(k,k+1)}) \\
&= \left(m_{k_{t_\ell}}, \beta^{(\ell)} f^{(k_{t_\ell})}, \beta^{(\ell)} f^{(k_{t_\ell},k_{t_{\ell+1}})} \circ_1 \beta^{(\ell,\ell+1)} f^{(k_{t_{\ell+1}})}\right).
\end{aligned}$$

where

$$\beta^{(\ell)} f^{(k_{t_\ell})} = g^{(t_\ell)} f^{\left(k_{t_\ell}\right)} = \alpha^{(\ell)}$$

and

$$\beta^{(\ell)} f^{(k_{t_\ell},k_{t_{\ell+1}})} \circ_1 \beta^{(\ell,\ell+1)} f^{(k_{t_{\ell+1}})} = g^{(t_\ell)} f^{(k_{t_\ell},k_{t_{\ell+1}})} \circ_1 g^{(t_\ell,t_{\ell+1})} f^{(k_{t_{\ell+1}})}$$

Let $L^{(t_\ell,t_{\ell+1})}$ be defined as in Lemma 3.10. Then we have that $L^{(t_\ell,t_{\ell+1})}$ is a 3-cell

$$\alpha^{(t_\ell,t_{\ell+1})} \Rrightarrow g^{(t_\ell)} f^{(k_{t_\ell},k_{t_{\ell+1}})} \circ_1 g^{(t_\ell,t_{\ell+1})} f^{(k_{t_{\ell+1}})}.$$

Hence, $(m_{k_{t_\ell}}, 1_{g^{(t_\ell)} f^{(k_{t_\ell})}}, L^{(t_\ell,t_{\ell+1})})$ is a 2-cell $h\,(gf) \Rightarrow (hg)\, f$. This concludes the proof. □

We now verify that composition of 1-cells preserves the relation of being connected by a 2-cell.

Lemma 3.12. Suppose that $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ are towers of complexes, $f = (m_k, f^{(k)}, f^{(k,k+1)})$, $f' = (m'_k, f'^{(k)}, f'^{(k,k+1)})$ are 1-cells from $\boldsymbol{A}$ to $\boldsymbol{B}$, and $h = \left(k_t, h^{(t)}, h^{(t,t+1)}\right)$ is a 1-cell from $\boldsymbol{B}$ to $\boldsymbol{C}$. Suppose that $m'_k \geq m_k$, $f'^{(k)} = f^{(k)} p^{\left(m_k, m'_k\right)}$, and $f'^{(k,k+1)} = f^{(k,k+1)} p^{(m_{k+1}, m'_{k+1})}$. Then there is a 2-cell $hf \Rightarrow hf'$.

Proof. By definition, we have that

$$hf = \left(m_{k_t}, h^{(t)} f^{(k_t)}, h^{(t)} f^{(k_t, k_{t+1})} \circ_1 h^{(t,t+1)} f^{(k_{t+1})}\right)$$

and

$$\begin{aligned} hf' &= \left(m'_{k_t}, h^{(t)} f'^{(k_t)}, h^{(t)} f'^{(k_t, k_{t+1})} \circ_1 h^{(t,t+1)} f'^{(k_{t+1})}\right) \\ &= \left(m'_{k_t}, h^{(t)} f^{(k_t)} p^{(m_{k_t}, m'_{k_t})}, h^{(t)} f^{(k_t, k_{t+1})} p^{(m_{k_{t+1}}, m'_{k_{t+1}})} \circ_1 h^{(t,t+1)} f^{(k_{t+1})} p^{(m_{k_{t+1}}, m'_{k_{t+1}})}\right) \end{aligned}$$

Furthermore,

$$\begin{aligned} & \left(h^{(t)} f'^{(k_t, k_{t+1})} \circ_1 h^{(t,t+1)} f'^{(k_{t+1})}\right) \circ_1 p^{(t,t+1)} 1_{h^{(t+1)} f'^{(k_{t+1})}} \\ =\; & \left(h^{(t)} f'^{(k_t, k_{t+1})} \circ_1 h^{(t,t+1)} f'^{(k_{t+1})}\right) \circ_1 1_{p^{(t,t+1)} h^{(t+1)} f'^{(k_{t+1})}} \\ =\; & \left(h^{(t)} f^{(k_t, k_{t+1})} p^{(m_{k_{t+1}}, m'_{k_{t+1}})} \circ_1 h^{(t,t+1)} f^{(k_{t+1})} p^{(m_{k_{t+1}}, m'_{k_{t+1}})}\right) \\ =\; & \left(h^{(t)} f^{(k_t, k_{t+1})} \circ_1 h^{(t,t+1)} f^{(k_{t+1})}\right) p^{(m_{k_{t+1}}, m'_{k_{t+1}})} \\ =\; & 1_{h^{(t)} f'^{(k_t)} p^{(m'_{k_t}, m'_{k_{t+1}})}} \circ_1 \left(h^{(t)} f^{(k_t, k_{t+1})} \circ_1 h^{(t,t+1)} f^{(k_{t+1})}\right) p^{(m_{k_t}, m'_{k_{t+1}})} \\ =\; & 1_{h^{(t)} f'^{(k_t)}} p^{(m'_{k_t}, m'_{k_{t+1}})} \circ_1 \left(h^{(t)} f^{(k_t, k_{t+1})} \circ_1 h^{(t,t+1)} f^{(k_{t+1})}\right) p^{(m_{k_t}, m'_{k_{t+1}})}. \end{aligned}$$

Therefore,

$$L := (m'_{k_t}, 1_{h^{(t)} f'^{(k_t)}}, 1_{h^{(t)} f'^{(k_t, k_{t+1})} \circ_1 h^{(t,t+1)} f'^{(k_{t+1})}})$$

is a 2-cell $hf \Rightarrow hf'$. □

Lemma 3.13. Suppose that $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ are towers of complexes, $f = (m_k, f^{(k)}, f^{(k,k+1)})$, $g = (m_k, g^{(k)}, g^{(k,k+1)})$ are 1-cells from $\boldsymbol{A}$ to $\boldsymbol{B}$, $h = \left(k_t, h^{(t)}, h^{(t,t+1)}\right)$ is a 1-cell from $\boldsymbol{B}$ to $\boldsymbol{C}$. Suppose that $L = \left(m_k, L^{(k)}, L^{(k,k+1)}\right)$ is a 2-cell from f to g. Then there exists a 2-cell $hf \Rightarrow hg$.

Proof. Define

$$hL = \left(m_{k_t}, S^{(t)}, S^{(t,t+1)}\right).$$

where

$$S^{(t)} = h^{(t)} L^{(k_t)}$$

and

$$S^{(t,t+1)} = H'^{(t,t+1)} \circ_2 H^{(t,t+1)}$$

for

$$H^{(t,t+1)} = 1_{h^{(t)} g^{(k_t, k_{t+1})}} \circ h^{(t,t+1)} L^{(k_{t+1})}$$

and

$$H'^{(t,t+1)} = h^{(t)} L^{(k,k+1)} \circ_1 1_{h^{(t,t+1)} f^{(k_{t+1})}}.$$

Notice that we have

$$L^{(k)} : f^{(k)} \Rightarrow g^{(k)}$$

and hence

$$S^{(t)} : h^{(t)} f^{(k_t)} \Rightarrow h^{(t)} g^{(k_t)}$$

Similarly, we have that

$$L^{(k,k+1)} : g^{(k,k+1)} \circ_1 p^{(k,k+1)} L^{(k+1)} \Rrightarrow L^{(k)} p^{(m_k, m_{k+1})} \circ_1 f^{(k,k+1)}$$

and hence

$$\begin{aligned}
& \left(h^{(t)}g^{(k_t,k_{t+1})} \circ_1 h^{(t,t+1)}f'^{(k_{t+1})}\right) \circ_1 p^{(k,k+1)}h^{(t+1)}L^{(k+1)} \\
= \quad & h^{(t)}g^{(k_t,k_{t+1})} \circ_1 h^{(t,t+1)}f'^{(k_{t+1})} \circ_1 p^{(t,t+1)}h^{(t+1)}L^{(k_{t+1})} \\
= \quad & h^{(t)}g^{(k_t,k_{t+1})} \circ_1 h^{(t,t+1)}t(L^{(k_{t+1})}) \circ_1 s(h^{(t,t+1)})L^{(k_{t+1})} \\
\overset{H^{(t,t+1)}}{\Rrightarrow} \quad & h^{(t)}g^{(k_t,k_{t+1})} \circ_1 t\left(h^{(t,t+1)}\right)L^{(k_{t+1})} \circ_1 h^{(t,t+1)}s(L^{(k_{t+1})}) \\
= \quad & h^{(t)}g^{(k_t,k_{t+1})} \circ_1 h^{(t)}p^{(k_t,k_{t+1})}L^{(k_{t+1})} \circ_1 h^{(t,t+1)}f^{(k_{t+1})}p^{(m_{k_{t+1}},m_{k_{t+1}})} \\
\overset{H'^{(t,t+1)}}{\Rrightarrow} \quad & h^{(t)}\left(L^{(k_t)}p^{(m_{k_t},m_{k_{t+1}})} \circ_1 f^{(k_t,k_{t+1})}\right) \circ_1 h^{(t,t+1)}f^{(k_{t+1})} \\
= \quad & h^{(t)}L^{(k_t)}p^{(m_{k_t},m_{k_{t+1}})} \circ_1 h^{(t)}f^{(k_t,k_{t+1})} \circ_1 h^{(t,t+1)}f^{(k_{t+1})} \\
= \quad & h^{(t)}L^{(k_t)}p^{(m_{k_t},m_{k_{t+1}})} \circ_1 \left(h^{(t)}f^{(k_t,k_{t+1})} \circ_1 h^{(t,t+1)}f^{(k_{t+1})}\right)
\end{aligned}$$

Thus, $S^{(t,t+1)} = H'^{(t,t+1)} \circ_2 H^{(t,t+1)}$ is a 3-cell from

$$\left(h^{(t)}g^{(k_t,k_{t+1})} \circ_1 h^{(t,t+1)}g^{(k_{t+1})}\right) \circ_1 p^{(k,k+1)}h^{(t+1)}L^{(k+1)}$$

to

$$h^{(t)}L^{(k_t)}p^{(m_{k_t},m_{k_{t+1}})} \circ_1 \left(h^{(t)}f^{(k_t,k_{t+1})} \circ_1 h^{(t,t+1)}f^{(k_{t+1})}\right).$$

This concludes the proof. ☐

Lemma 3.14. Suppose that $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ are towers of complexes, $f = (m_k, f^{(k)}, f^{(k,k+1)})$, $g = (n_k, g^{(k)}, g^{(k,k+1)})$ are 1-cells from $\boldsymbol{A}$ to $\boldsymbol{B}$, $h = \left(k_t, h^{(t)}, h^{(t,t+1)}\right)$ is a 1-cell from $\boldsymbol{B}$ to $\boldsymbol{C}$. If there is a 2-cell $f \Rightarrow g$, then there is a 2-cell $hf \Rightarrow hg$.

Proof. Let $L = \left(\ell_k, L^{(k)}, L^{(k,k+1)}\right)$ be a 2-cell from f to g. Define

$$f' = \left(\ell_k, f^{(k)}p^{(m_k,\ell_k)}, f^{(k,k+1)}p^{(m_{k+1},\ell_{k+1})}\right)$$

and

$$g' = \left(\ell_k, g^{(k)}p^{(m_k,\ell_k)}, g^{(k,k+1)}p^{(m_{k+1},g_{k+1})}\right).$$

Then by Lemma 3.12 and Lemma 3.13 we have that there exist 2-cells

$$hf \Rightarrow hf' \Rightarrow hg' \Rightarrow hg.$$

By Lemma 3.9, this concludes the proof. ☐

Lemma 3.15. Suppose that $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ are towers of complexes, $f = (m_k, f^{(k)}, f^{(k,k+1)})$ is a 1-cell from $\boldsymbol{A}$ to $\boldsymbol{B}$, $g = \left(k_t, g^{(t)}, h^{(t,t+1)}\right)$ and $g' = \left(k'_t, g'^{(t)}, g'^{(t,t+1)}\right)$ are 1-cells from $\boldsymbol{B}$ to $\boldsymbol{C}$, such that $k'_t \geq k_t$ for $t \in \omega$, $g'^{(t)} = g^{(t)}p^{(k_t,k'_t)}$, and $g'^{(t,t+1)} = g^{(t,t+1)}p^{(k_{t+1},k'_{t+1})}$. Then there exists a 2-cell $g'f \Rightarrow gf$.

Proof. Define

$$L = (m_{k'_t}, S^{(t)}, S^{(t,t+1)})$$

where

$$S^{(t)} = g^{(t)}f^{(k_t,k'_t)}$$

and

$$S^{(t,t+1)} = 1_{g^{(t)}f^{(k_t,k_{t+1})}p^{(m_{k_{t+1}},m_{k'_{t+1}})}} \circ_1 g^{(t,t+1)}f^{(k_{t+1},k'_{t+1})}$$

We claim that $L : g'f \Rightarrow gf$ is a 2-cell.

We have that

$$gf = \left(m_{k_t}, g^{(t)}f^{(k_t)}, g^{(t)}f^{(k_t,k_{t+1})} \circ_1 g^{(t,t+1)}f^{(k_{t+1})}\right)$$

and

$$\begin{aligned} g'f &= \left(m_{k'_t}, g'^{(t)}f^{\left(k'_t\right)}, g'^{(t)}f^{\left(k'_t,k'_{t+1}\right)} \circ_1 g'^{(t,t+1)}f^{\left(k'_{t+1}\right)}\right) \\ &= \left(m_{k'_t}, g^{(t)}p^{(k_t,k'_t)}f^{(k'_t)}, g^{(t)}p^{(k_t,k'_t)}f^{\left(k'_t,k'_{t+1}\right)} \circ_1 g^{(t,t+1)}p^{(k_{t+1},k'_{t+1})}f^{\left(k'_{t+1}\right)}\right). \end{aligned}$$

We have that

$$f^{\left(k_t,k'_t\right)} : p^{\left(k_t,k'_t\right)}f^{\left(k'_t\right)} \Rightarrow f^{(k_t)}p^{(m_{k_t},m_{k'_t})}.$$

Therefore

$$g^{(t)}f^{(k_t,k'_t)} : g'^{(t)}f^{\left(k'_t\right)} \Rightarrow g^{(t)}f^{(k_t)}p^{(m_{k_t},m_{k'_t})}.$$

Furthermore, we have that

$$\begin{aligned} & \left(g^{(t)}f^{(k_t,k_{t+1})} \circ_1 g^{(t,t+1)}f^{(k_{t+1})}\right)p^{\left(m_{k_{t+1}},m_{k'_{t+1}}\right)} \circ_1 p^{(t,t+1)}S^{(t+1)} \\ = & \left(g^{(t)}f^{(k_t,k_{t+1})} \circ_1 g^{(t,t+1)}f^{(k_{t+1})}\right)p^{\left(m_{k_{t+1}},m_{k'_{t+1}}\right)} \circ_1 p^{(t,t+1)}g^{(t+1)}f^{(k_{t+1},k'_{t+1})} \\ = & \; g^{(t)}f^{(k_t,k_{t+1})}p^{\left(m_{k_{t+1}},m_{k'_{t+1}}\right)} \circ_1 g^{(t,t+1)}f^{(k_{t+1})}p^{\left(m_{k_{t+1}},m_{k'_{t+1}}\right)} \circ_1 p^{(t,t+1)}g^{(t+1)}f^{(k_{t+1},k'_{t+1})} \\ = & \; g^{(t)}f^{(k_t,k_{t+1})}p^{\left(m_{k_{t+1}},m_{k'_{t+1}}\right)} \circ_1 g^{(t,t+1)}t\left(f^{(k_{t+1},k'_{t+1})}\right) \circ_1 s\left(g^{(t,t+1)}\right)f^{(k_{t+1},k'_{t+1})} \\ \overset{S^{(t,t+1)}}{\Rrightarrow} & \; g^{(t)}f^{(k_t,k_{t+1})}p^{\left(m_{k_{t+1}},m_{k'_{t+1}}\right)} \circ_1 t\left(g^{(t,t+1)}\right)f^{(k_{t+1},k'_{t+1})} \circ_1 g^{(t,t+1)}s\left(f^{\left(k_{t+1},k'_{t+1}\right)}\right) \\ = & \; g^{(t)}f^{(k_t,k_{t+1})}p^{\left(m_{k_{t+1}},m_{k'_{t+1}}\right)} \circ g^{(t)}p^{(k_t,k_{t+1})}f^{(k_{t+1},k'_{t+1})} \circ_1 g^{(t,t+1)}p^{(k_{t+1},k'_{t+1})}f^{\left(k'_{t+1}\right)} \\ = & \; S^{(t)}p^{\left(m_{k'_t},m_{k'_{t+1}}\right)} \circ_1 \left(g^{(t)}p^{(k_t,k'_t)}f^{\left(k'_t,k'_{t+1}\right)} \circ_1 g^{(t,t+1)}p^{(k_{t+1},k'_{t+1})}f^{\left(k'_{t+1}\right)}\right). \end{aligned}$$

This shows that

$$S^{(t,t+1)} = 1_{g^{(t)}f^{(k_t,k_{t+1})}p^{\left(m_{k_{t+1}},m_{k'_{t+1}}\right)}} \circ_1 g^{(t,t+1)}f^{(k_{t+1},k'_{t+1})}$$

is a 3-cell from

$$\left(g^{(t)}f^{(k_t,k_{t+1})} \circ_1 g^{(t,t+1)}f^{(k_{t+1})}\right)p^{\left(m_{k_{t+1}},m_{k'_{t+1}}\right)} \circ_1 p^{(t,t+1)}S^{(t+1)}$$

to

$$S^{(t)}p^{\left(m_{k'_t},m_{k'_{t+1}}\right)} \circ_1 \left(g^{(t)}p^{(k_t,k'_t)}f^{\left(k'_t,k'_{t+1}\right)} \circ_1 g^{(t,t+1)}p^{(k_{t+1},k'_{t+1})}f^{\left(k'_{t+1}\right)}\right).$$

This concludes the proof $L : g'f \Rightarrow gf$ is a 2-cell. □

Lemma 3.16. Suppose that $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ are towers of complexes, $f = (m_k, h^{(k)}, h^{(k,k+1)})$ is a 1-cell from $\boldsymbol{A}$ to $\boldsymbol{B}$, $g = \left(k_t, g^{(t)}, g^{(t,t+1)}\right)$ and $h = \left(k_t, h^{(t)}, h^{(t,t+1)}\right)$ are 1-cells from $\boldsymbol{B}$ to $\boldsymbol{C}$, and $L = \left(k_t, L^{(t)}, L^{(t,t+1)}\right)$ is a 2-cell from g to h. Then there exists a 2-cell $gf \Rightarrow hf$.

Proof. Define

$$Lf = \left(m_{k_t}, S^{(t)}, S^{(t,t+1)}\right)$$

where

$$S^{(t)} = L^{(t)}f^{(k_t)}$$

and

$$S^{(t,t+1)} = H'^{(t,t+1)} \circ_2 H^{(t,t+1)}$$

for

$$H'^{(t,t+1)} = -\left(L^{(t)}f^{(k_t,k_{t+1})} \circ_1 1_{g^{(t,t+1)}f^{(k_{t+1})}}\right)$$

and

$$H^{(t,t+1)} = 1_{h^{(t)}f^{(k_t,k_{t+1})}} \circ_1 L^{(t,t+1)}f^{(k_{t+1})}.$$

We claim that $Lf : gf \Rightarrow hf$ is a 2-cell.

We have that

$$L^{(t)} : g^{(t)} \Rightarrow h^{(t)}$$

and hence

$$L^{(t)} f^{(k_t)} : g^{(t)} f^{(k_t)} \Rightarrow h^{(t)} f^{(k_t)}$$

Similarly, we have that

$$L^{(t,t+1)} : h^{(t,t+1)} \circ_1 p^{(t,t+1)} L^{(t+1)} \Rrightarrow L^{(t)} p^{(k_t,k_{t+1})} \circ_1 g^{(t,t+1)}$$

Thus, we have that

$$L^{(t)} f^{(k_t)} : g^{(t)} f^{(k_t)} \Rightarrow h^{(t)} f^{(k_t)}.$$

Furthermore, we have that

$$\begin{array}{rl} & \left(h^{(t)} f^{(k_t,k_{t+1})} \circ_1 h^{(t,t+1)} f^{(k_{t+1})}\right) \circ_1 p^{(t,t+1)} L^{(t+1)} f^{(k_{t+1})} \\ = & h^{(t)} f^{(k_t,k_{t+1})} \circ_1 \left(h^{(t,t+1)} \circ_1 p^{(t,t+1)} L^{(t+1)}\right) f^{(k_{t+1})} \\ \overset{H^{(t,t+1)}}{\Rrightarrow} & h^{(t)} f^{(k_t,k_{t+1})} \circ_1 \left(L^{(t)} p^{(k_t,k_{t+1})} \circ_1 g^{(t,t+1)}\right) f^{(k_{t+1})} \\ = & h^{(t)} f^{(k_t,k_{t+1})} \circ_1 L^{(t)} p^{(k_t,k_{t+1})} f^{(k_{t+1})} \circ_1 g^{(t,t+1)} f^{(k_{t+1})} \\ = & t\left(L^{(t)}\right) f^{(k_t,k_{t+1})} \circ_1 L^{(t)} s\left(f^{(k_t,k_{t+1})}\right) \circ_1 g^{(t,t+1)} f^{(k_{t+1})} \\ \overset{H'^{(t,t+1)}}{\Rrightarrow} & L^{(t)} t\left(f^{(k_t,k_{t+1})}\right) \circ_1 s\left(L^{(t)}\right) f^{(k_t,k_{t+1})} \circ_1 g^{(t,t+1)} f^{(k_{t+1})} \\ = & L^{(t)} f^{(k_t)} p^{(m_{k_t},m_{k_{t+1}})} \circ_1 g^{(t)} f^{(k_t,k_{t+1})} \circ_1 g^{(t,t+1)} f^{(k_{t+1})} \end{array}$$

This shows that $S^{(t,t+1)} = H'^{(t,t+1)} \circ_2 H^{(t,t+1)}$ is a 3-cell from

$$\left(h^{(t)} f^{(k_t,k_{t+1})} \circ_1 h^{(t,t+1)} f^{(k_{t+1})}\right) \circ_1 p^{(t,t+1)} L^{(t+1)} f^{(k_{t+1})}$$

to

$$L^{(t)} f^{(k_t)} p^{(m_{k_t},m_{k_{t+1}})} \circ_1 \left(g^{(t)} f^{(k_t,k_{t+1})} \circ_1 g^{(t,t+1)} f^{(k_{t+1})}\right).$$

This concludes the proof that $Lf : gf \Rightarrow hf$ is a 2-cell. ☐

Lemma 3.17. Suppose that $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ are towers of complexes, $f = (m_k, h^{(k)}, h^{(k,k+1)})$ is a 1-cell from $\boldsymbol{A}$ to $\boldsymbol{B}$, $g = \left(n_t, g^{(t)}, g^{(t,t+1)}\right)$ and $h = \left(k_t, h^{(t)}, h^{(t,t+1)}\right)$ are 1-cells from $\boldsymbol{B}$ to $\boldsymbol{C}$. If there exists a 2-cell $g \Rightarrow h$, then there exists a 2-cell $gf \Rightarrow hf$.

Proof. Suppose that $\left(\ell_t, L^{(t)}, L^{(t,t+1)}\right)$ is a 2-cell from g to h. Define

$$g' = \left(\ell_t, g^{(t)} p^{(n_t,\ell_t)}, g^{(t,t+1)} p^{(n_{t+1},\ell_{t+1})}\right)$$

and

$$h' = \left(\ell_t, h^{(t)} p^{(k_t,\ell_t)}, h^{(t,t+1)} p^{(k_{t+1},\ell_{t+1})}\right).$$

Then by Lemma 3.15 and Lemma 3.16 there exist 2-cells

$$gf \Rightarrow g'f \Rightarrow h'f \Rightarrow hf.$$

By Lemma 3.9, this concludes the proof. ☐

We can finally present the proof of Theorem 3.6.

Proof of Theorem 3.6. By Lemma 3.9, the relation among 1-cells defined by being connected by 2-cells is indeed an equivalence relation. Lemma 3.17 and Lemma 3.14 show that the operation of composition of coherent morphisms is well-defined. Such an operation is associative by Lemma 3.11. ☐

4. The homotopy limit of a tower of complexes

Suppose now that R is a ring, and $\mathcal{A}$ is an additive category of R-modules that is closed under *countable products* and *equalizers*. We associate as in [4, Section 17] with a tower of complexes $\boldsymbol{A}$ a complex $\mathrm{holim}\boldsymbol{A}$, called its *homotopy limit*.

For $n \in \mathbb{Z}$, define $C_n(\boldsymbol{A}) = (\mathrm{holim}\boldsymbol{A})_n$ to be the submodule of

$$\prod_{m\in\omega} A_n^{(m)} \oplus \prod_{m_0\leq m_1} A_{n+1}^{(m_0)}$$

consisting of those elements $z = (z_m, z_{m_0,m_1})$ such that

$$x_{m_0m_1} + p_n^{(m_0,m_1)}(x_{m_1,m_2}) = x_{m_0m_2}$$

for every $m_0 \leq m_1 \leq m_2$. The differential

$$d_n : C_{n+1}(\boldsymbol{A}) \to C_n(\boldsymbol{A})$$

is defined by setting, for $z \in C_n(\boldsymbol{A})$,

$$(d_n z)_m = \partial_n z_m$$

and

$$(d_n z)_{m_0,m_1} = \partial_{n+1} z_{m_0,m_1} + (-1)^n \left(p^{(m_0,m_1)}(z_{m_1}) - z_{m_0}\right)$$

for every $m \in \omega$ and $m_0 \leq m_1$. This defines a complex $\mathrm{holim}\boldsymbol{A}$ in $\mathcal{A}$.

Remark 4.1. We consider here only pairs of indices, since we are restricting to *towers*. When considering arbitrary inverse sequences, one should consider arbitrary finite increasing tuples of indices as in [4, Section 17].

Remark 4.2. If $z \in C_n(\boldsymbol{A})$, then one has that

$$z_{m_0,m_0} = 0$$

and

$$z_{m_0,m_1} = \sum_{m=m_0}^{m_1-1} p^{(m_0,m)}(z_{m,m+1})$$

for every $m_0 < m_1$. Thus the values $z_{m,m+1}$ for $m \in \omega$ determine the values z_{m_0,m_1} for every $m_0 \leq m_1$. This observation will be tacitly used in the following.

Suppose that $\boldsymbol{A}$ and $\boldsymbol{B}$ are towers of complexes, and $f = (m_k, f^{(k)}, f^{(k,k+1)})$ is a 1-cell from $\boldsymbol{A}$ to $\boldsymbol{B}$. We define a chain map $f^{(\infty)} = \mathrm{holim}f$ from $\mathrm{holim}\boldsymbol{A}$ to $\mathrm{holim}\boldsymbol{B}$, called the (homotopy) *limit* 1-cell of f, as follows. For $n \in \mathbb{Z}$, $z \in C_n(\boldsymbol{A})$, we define $f_n^{(\infty)}(z) \in C_n(\boldsymbol{B})$, by setting

$$f_n^{(\infty)}(z)_k = f_n^{(k)}(z_{m_k})$$

and

$$f_n^{(\infty)}(z)_{k,k+1} = f_{n+1}^{(k)}(z_{m_k,m_{k+1}}) + (-1)^n f_n^{(k,k+1)}(z_{m_{k+1}})$$

for every $k \in \omega$. This definition can be seen as the algebraic analogue of the chain map induced by a coherent map as in [4, Section 18]. The rest of this section is dedicated to the proof of the following theorem.

Theorem 4.3. *Suppose that R is a ring, and $\mathcal{A}$ is an additive category of R-modules that is closed under countable products and equalizers. Then the assignment $\boldsymbol{A} \mapsto \mathrm{holim}\boldsymbol{A}$ and $[f] \mapsto \mathrm{holim}[f] := [\mathrm{holim}f]$ defines a functor from the coherent category of towers of complexes in $\mathcal{A}$ to the homotopy category of complexes in $\mathcal{A}$.*

Towards a proof of Theorem 4.3, we begin with verifying that $\mathrm{holim}f$ is indeed a chain map. The proof of the following lemma can be seen as an algebraic analogue of [4, Lemma 18.3] (or its version for 1-coherent mappings).

Lemma 4.4. Adopt the notations above. The sequence $(f_n^{(\infty)})_{n\in\mathbb{Z}}$ defines a chain map $f^{(\infty)}$ from $\mathrm{holim}\boldsymbol{A}$ to $\mathrm{holim}\boldsymbol{B}$.

Proof. We have that, for $n \in \mathbb{Z}$, $z \in C_n(\boldsymbol{A})$, and $k \in \omega$,

$$\begin{aligned}\left(d_n f_n^{(\infty)}\right)(z)_k &= \partial(f_n^{(\infty)}(z)_k) = \partial(f_n^{(k)}(z_{m_k})) \\ &= f_n^{(k)}(\partial z_{m_k}) = f_n^{(k)}(d_n(z)_{m_k}) = \left(f_n^{(\infty)} d_n\right)(z)_k\end{aligned}$$

Similarly, we have that

$$\begin{aligned}&\left(d_n f_n^{(\infty)}\right)(z)_{k,k+1} \\ =\ & \partial f_n^{(\infty)}(z)_{k,k+1} + (-1)^n \left(p^{(k,k+1)}\left(f_n^{(\infty)}(z)_{k+1}\right) - f_n^{(\infty)}(z)_k\right) \\ =\ & \partial\left(f_{n+1}^{(k)}(z_{m_k,m_{k+1}}) + (-1)^n f_n^{(k,k+1)}\left(z_{m_{k+1}}\right)\right) \\ & + (-1)^n \left(p^{(k,k+1)} f_n^{(k+1)}\left(z_{m_{k+1}}\right) - f_n^{(k)}\left(z_{m_k}\right)\right) \\ =\ & f_n^{(k)}\left(\partial z_{m_k,m_{k+1}}\right) \\ & + (-1)^n \left(-f_{n-1}^{(k,k+1)}\partial + f_n^{(k)} p^{(m_k,m_{k+1})} - p^{(k,k+1)} f_n^{(k+1)}\right)\left(z_{m_{k+1}}\right) \\ & + (-1)^n \left(p^{(k,k+1)} f_n^{(k+1)}\left(z_{m_{k+1}}\right) - f_n^{(k)}\left(z_{m_k}\right)\right) \\ =\ & f_n^{(k)}\left(\partial z_{m_k,m_{k+1}}\right) \\ & + (-1)^n \left(-f_{n-1}^{(k,k+1)}\partial + f_n^{(k)} p^{(m_k,m_{k+1})}\right)\left(z_{m_{k+1}}\right) + (-1)^n \left(-f_n^{(k)}\left(z_{m_k}\right)\right) \\ =\ & f_n^{(k)}\left(\partial z_{m_k,m_{k+1}}\right) + (-1)^{n-1} f_{n-1}^{(k,k+1)}\partial\left(z_{m_{k+1}}\right) \\ & + f_n^{(k)}\left((-1)^n \left(p^{(m_k,m_{k+1})}\left(z_{m_{k+1}}\right) - \left(z_{m_k}\right)\right)\right) \\ =\ & f_n^{(k)}\left(\partial z_{m_k,m_{k+1}} + (-1)^n \left(p^{(m_k,m_{k+1})}\left(z_{m_{k+1}}\right) - \left(z_{m_k}\right)\right)\right) + (-1)^{n-1} f_{n-1}^{(k,k+1)}\left(\partial z_{m_{k+1}}\right) \\ =\ & f_n^{(k)}\left(d_n(z)_{k,k+1}\right) + (-1)^n f_{n-1}^{(k,k+1)}\left(d_n(z)_{k+1}\right) \\ =\ & \left(f_{n-1}^{(\infty)} d_n\right)(z)_{k,k+1}.\end{aligned}$$

This concludes the proof that $f^{(\infty)}$ is a chain map from $\boldsymbol{A}$ to $\boldsymbol{B}$. ☐

Our goal is to show that the chain homotopy class of $\mathrm{holim} f$ only depends on the equivalence class of f up to 2-cells. The following lemma can be seen as the analogue in our context of [4, Lemma 18.4].

Lemma 4.5. Suppose that $\boldsymbol{A}$ and $\boldsymbol{B}$ are towers of complexes, and that $f = (m_k, f^{(k)}, f^{(k,k+1)})$ and $f' = (\ell_k, f'^{(k)}, f'^{(k,k+1)})$ are 1-cells from $\boldsymbol{A}$ to $\boldsymbol{B}$ such that $\ell_k \geq m_k$, $f'^{(k)} = f^{(k)} p^{(m_k,\ell_k)}$ and $f'^{(k,k+1)} = f^{(k,k+1)} p^{(m_{k+1},\ell_{k+1})}$ for every $k \in \omega$. Then there is a chain homotopy $f^{(\infty)} \Rightarrow f'^{(\infty)}$.

Proof. Fix $n \in \mathbb{Z}$. Define $F_n^{(\infty)} : C_n(\boldsymbol{A}) \to C_{n+1}(\boldsymbol{B})$ by setting, for $z \in C_n(\boldsymbol{A})$ and $k \in \omega$,

$$F_n^{(\infty)}(z)_k = (-1)^{n+1} f_{n+1}^{(k)}(z_{m_k,\ell_k})$$

and

$$F_n^{(\infty)}(z)_{k,k+1} = f_{n+1}^{(k,k+1)}(z_{m_{k+1},\ell_{k+1}}).$$

We claim that $F^{(\infty)} = (F_n^{(\infty)})_{n \in \mathbb{Z}}$ is a chain homotopy $f^{(\infty)} \Rightarrow f'^{(\infty)}$.

We have that, for every $n \in \mathbb{Z}$, $z \in C_n(\boldsymbol{A})$, and $k \in \omega$,,

$$\begin{aligned}(d_{n+1} F_n^{(\infty)}(z))_k &= \partial(F_n^{(\infty)}(z)_k) \\ &= (-1)^{n+1} \partial(f_{n+1}^{(k)}(z_{m_k,\ell_k})) \\ &= (-1)^{n+1} f_n^{(k)}(\partial z_{m_k,\ell_k})\end{aligned}$$

and

$$\begin{aligned}(F^{(\infty)}_{n-1}d_n)(z)_k &= f^{(k)}_n(d_n(z)_{m_k,\ell_k}) \\ &= (-1)^n\left(f^{(k)}_n(\partial z_{m_k,\ell_k} + (-1)^n\left(p^{(m_k,\ell_k)}(z_{\ell_k}) - z_{m_k}\right)\right) \\ &= f^{(k)}_n p^{(m_k,\ell_k)}(z_{\ell_k}) - f^{(k)}_n(z_{m_k}) + (-1)^n f^{(k)}_n\partial(z_{m_k,\ell_k})\end{aligned}$$

Therefore

$$\begin{aligned}(d_{n+1}F^{(\infty)}_n + F^{(\infty)}_{n-1}d_n)(z)_k &= f^{(k)}_n p^{(m_k,\ell_k)}(z_{\ell_k}) - f^{(k)}_n(z_{m_k}) \\ &= f'^{(k)}_n(z_{\ell_k}) - f^{(k)}_n(z_{m_k}) \\ &= \left(f'^{(\infty)} - f^{(\infty)}\right)(z)_k\,.\end{aligned}$$

Similarly,

$$\begin{aligned}&(d_{n+1}F^{(\infty)}_n)(z)_{k,k+1} \\ =\;& \partial(F^{(\infty)}_n(z)_{k,k+1}) + (-1)^{n+1}\left(p^{(k,k+1)}F^{(\infty)}_n(z)_{k+1} - F^{(\infty)}_n(z)_k\right) \\ =\;& \partial f^{(k,k+1)}_{n+1}(z_{m_{k+1},\ell_{k+1}}) + p^{(k,k+1)}f^{(k+1)}_{n+1}(z_{m_{k+1},\ell_{k+1}}) - f^{(k)}_{n+1}(z_{m_k,\ell_k})\end{aligned}$$

and

$$\begin{aligned}&(F^{(\infty)}_{n-1}d_n)\,(z)_{k,k+1} \\ =\;& f^{(k,k+1)}_n(d_n\,(z)_{m_{k+1},\ell_{k+1}}) \\ =\;& f^{(k,k+1)}_n(\partial z_{m_{k+1},\ell_{k+1}} + (-1)^n\left(p^{(m_{k+1},\ell_{k+1})}(z_{\ell_{k+1}}) - z_{m_{k+1}}\right) \\ =\;& (f^{(k,k+1)}_n\partial)(z_{m_{k+1},\ell_{k+1}}) + (-1)^n f^{(k,k+1)}_n p^{(m_{k+1},\ell_{k+1})}(z_{\ell_{k+1}}) + (-1)^{n+1} f^{(k,k+1)}_n(z_{m_{k+1}}).\end{aligned}$$

Since $f^{(k,k+1)} : p^{(k,k+1)}f^{(k+1)} \Rightarrow p^{(m_k,m_{k+1})}f^{(k)}$ is a 2-cell, we have that

$$f^{(k,k+1)}_n\partial + \partial f^{(k,k+1)}_{n+1} = f^{(k)}_{n+1}p^{(m_k,m_{k+1})} - p^{(k,k+1)}f^{(k+1)}_{n+1}.$$

We can thus conclude that

$$\begin{aligned}&(d_{n+1}F^{(\infty)}_n + F^{(\infty)}_{n-1}d_n)\,(z)_{k,k+1} \\ =\;& (\partial f^{(k,k+1)}_{n+1} + f^{(k,k+1)}_n\partial)(z_{m_{k+1},\ell_{k+1}}) + p^{(k,k+1)}f^{(k+1)}_{n+1}(z_{m_{k+1},\ell_{k+1}}) - f^{(k)}_{n+1}(z_{m_k,\ell_k}) \\ &+ (-1)^n f^{(k,k+1)}_n p^{(m_{k+1},\ell_{k+1})}(z_{\ell_{k+1}}) + (-1)^{n+1} f^{(k,k+1)}_n(z_{m_{k+1}}) \\ =\;& f^{(k)}_{n+1}\left(p^{(m_k,m_{k+1})}(z_{m_{k+1},\ell_{k+1}}) - z_{m_k,\ell_k}\right) + (-1)^n f^{(k,k+1)}_n\left(p^{(m_{k+1},\ell_{k+1})}(z_{\ell_{k+1}}) - z_{m_{k+1}}\right)\end{aligned}$$

We also have that

$$\begin{aligned}&\left(f'^{(\infty)} - f^{(\infty)}\right)(z)_{k,k+1} \\ =\;& f'^{(k)}_{n+1}(z_{\ell_k,\ell_{k+1}}) + (-1)^n f'^{(k,k+1)}_n(z_{\ell_{k+1}}) \\ &- f^{(k)}_{n+1}(z_{m_k,m_{k+1}}) + (-1)^{n+1} f^{(k,k+1)}_n(z_{m_{k+1}}) \\ =\;& f^{(k)}_{n+1}p^{(m_k,\ell_k)}\left(z_{\ell_k,\ell_{k+1}}\right) + (-1)^n f^{(k,k+1)}_n p^{(m_{k+1},\ell_{k+1})}\left(z_{\ell_{k+1}}\right) \\ &- f^{(k)}_{n+1}(z_{m_k,m_{k+1}}) + (-1)^{n+1} f^{(k,k+1)}_n(z_{m_{k+1}}) \\ =\;& f^{(k)}_{n+1}\left(p^{(m_k,\ell_k)}\left(z_{\ell_k,\ell_{k+1}}\right) - z_{m_k,m_{k+1}}\right) + (-1)^n f^{(k,k+1)}_n\left(p^{(m_{k+1},\ell_{k+1})}\left(z_{\ell_{k+1}}\right) - z_{m_{k+1}}\right)\end{aligned}$$

Notice now that

$$\begin{aligned}&\left(p^{(m_k,m_{k+1})}(z_{m_{k+1},\ell_{k+1}}) - z_{m_k,\ell_k}\right) - \left(p^{(m_k,\ell_k)}\left(z_{\ell_k,\ell_{k+1}}\right) - z_{m_k,m_{k+1}}\right) \\ =\;& \left(p^{(m_k,m_{k+1})}(z_{m_{k+1},\ell_{k+1}}) + z_{m_k,m_{k+1}}\right) - \left(p^{(m_k,\ell_k)}\left(z_{\ell_k,\ell_{k+1}}\right) + z_{m_k,\ell_k}\right) \\ =\;& z_{m_k,\ell_{k+1}} - z_{m_k,\ell_{k+1}} = 0\end{aligned}$$

This shows that
$$\left(d_{n+1}F_n^{(\infty)}+F_{n-1}^{(\infty)}d_n\right)(z)_{k,k+1}=\left(f'^{(\infty)}-f^{(\infty)}\right)(z)_{k,k+1}$$
concluding the proof. □

Lemma 4.6. Suppose that $\boldsymbol{A}$ and $\boldsymbol{B}$ are towers of complexes, and that $f=(\ell_k,f^{(k)},f^{(k,k+1)})$ and $g=(\ell_k,g^{(k)},g^{(k,k+1)})$ are 1-cells from $\boldsymbol{A}$ to $\boldsymbol{B}$ and $L=(\ell_k,L^{(k)},L^{(k,k+1)})_{k\in\omega}$ is a 2-cell from f to g. Then there is a chain homotopy $f^{(\infty)}\Rightarrow g^{(\infty)}$.

Proof. Fix $n\in\mathbb{Z}$. Define $L_n^{(\infty)}:C_n(\boldsymbol{A})\to C_{n+1}(\boldsymbol{B})$ by setting, for $z\in C_n(\boldsymbol{A})$ and $k\in\omega$,
$$L_n^{(\infty)}(z)_k=L_n^{(k)}(z_{\ell_k})$$
and
$$L_n^{(\infty)}(z)_{k,k+1}=L_{n+1}^{(k)}(z_{\ell_k,\ell_{k+1}})+(-1)^{n+1}L_n^{(k,k+1)}(z_{\ell_{k+1}}).$$
We claim that the sequence $(L_n^{(\infty)})_{n\in\mathbb{Z}}$ defines a chain homotopy $f^{(\infty)}\Rightarrow g^{(\infty)}$.

For $n\in\mathbb{Z}$, $z\in C_n(\boldsymbol{A})$, and $k\in\omega$, we have that
$$\begin{aligned}
&\left(d_{n+1}L_n^{(\infty)}+L_{n-1}^{(\infty)}d_n\right)(z)_k\\
=\;&\partial(L_n^{(\infty)}(z)_k)+L_{n-1}(d(z)_k)\\
=\;&\partial L_n^{(k)}(z_{\ell_k})+L_{n-1}^{(k)}\partial(z_{\ell_k})\\
=\;&(\partial L_n^{(k)}+L_{n-1}^{(k)}\partial)(z_{\ell_k})\\
=\;&(g_n^{(k)}p^{(n_k,\ell_k)}-f_n^{(k)}p^{(m_k,\ell_k)})(z_{\ell_k}).
\end{aligned}$$
Similarly, we have that
$$\begin{aligned}
&\left(d_{n+1}L_n^{(\infty)}+L_{n-1}^{(\infty)}d_n\right)(z)_{k,k+1}\\
=\;&\partial(L_n(z)_{k,k+1})+(-1)^{n+1}(p^{(k,k+1)}L_n(z)_{k+1}-L_n(z)_k)\\
&+L_n^{(k)}(d_n(z)_{m_k,m_{k+1}})+(-1)^nL_{n-1}^{(k,k+1)}(d_n(z)_{m_{k+1}})\\
=\;&\partial\left(L_{n+1}^{(k)}(z_{\ell_k,\ell_{k+1}})+(-1)^{n+1}L_n^{(k,k+1)}(z_{\ell_{k+1}})\right)\\
&+(-1)^{n+1}\left(p^{(k,k+1)}L_n^{(k+1)}(z_{k+1})-L_n^{(k)}(z_k)\right)\\
&+L_n^{(k)}\left(\partial z_{\ell_k,\ell_{k+1}}+(-1)^n(p^{(\ell_k,\ell_{k+1})}(z_{\ell_{k+1}})-z_{\ell_k})\right)+(-1)^nL_{n-1}^{(k,k+1)}(\partial z_{\ell_{k+1}})\\
=\;&\left(\partial L_{n+1}^{(k)}+L_n^{(k)}\partial\right)(z_{\ell_k,\ell_{k+1}})\\
&+(-1)^{n+1}\left(\partial L_n^{(k,k+1)}-L_{n-1}^{(k,k+1)}\partial+p^{(k,k+1)}L_n^{(k+1)}-L_n^{(k)}p^{(\ell_k,\ell_{k+1})}\right)(z_{\ell_{k+1}})\\
=\;&\left(g_{n+1}^{(k)}-f_{n+1}^{(k)}\right)(z_{\ell_k,\ell_{k+1}})\\
&+(-1)^{n+1}\left(f_n^{(k,k+1)}-g_n^{(k,k+1)}\right)(z_{\ell_{k+1}})\\
=\;&\left(g^{(\infty)}-f^{(\infty)}\right)(z)_{k,k+1}.
\end{aligned}$$
This concludes the proof. □

The following lemma is the algebraic analogue of [4, Lemma 18.6].

Lemma 4.7. Suppose that $\boldsymbol{A}$ and $\boldsymbol{B}$ are towers of complexes, $f=(m_k,f^{(k)},f^{(k,k+1)})$ and $g=(n_k,g^{(k)},g^{(k,k+1)})$ are 1-cells from $\boldsymbol{A}$ to $\boldsymbol{B}$. If there exists a 2-cell $f\Rightarrow g$, then there exists a chain homotopy $f^{(\infty)}\Rightarrow g^{(\infty)}$.

Proof. Suppose that $L=(\ell_k,L^{(k)},L^{(k,k+1)})_{k\in\omega}$ is a 2-cell from f to g. Define the 1-cells
$$f'=\left(\ell_k,f^{(k)}p^{(m_k,\ell_k)},f^{(k,k+1)}p^{(m_{k+1},\ell_{k+1})}\right)$$

and

$$g' = \left(\ell_k, g^{(k)} p^{(n_k,\ell_k)}, g^{(k,k+1)} p^{(n_{k+1},\ell_{k+1})}\right)$$

from $\boldsymbol{A}$ to $\boldsymbol{B}$. By Lemma 4.5 and Lemma 4.6, there exist chain homotopies

$$f^{(\infty)} \Rightarrow f'^{(\infty)} \Rightarrow g'^{(\infty)} \Rightarrow g^{(\infty)}.$$

This concludes the proof. □

Lemma 4.8. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be towers of complexes. Suppose that $(m_k)_{k\in\omega}$ is an increasing sequence in ω. Define $\boldsymbol{A}'$ to be the tower $\left(A^{(m_k)}\right)_{k\in\omega}$ and $\boldsymbol{B}'$ to be the tower $\left(B^{(m_k)}\right)_{k\in\omega}$. Let $f = \left(m_k, f^{(k)}, f^{(k,k+1)}\right)$ be a 1-cell $\boldsymbol{A}$ to $\boldsymbol{B}$. Notice that f induces a level 1-cell $f' = \left(k, f^{(m_k)}, f^{(m_k,m_{k+1})}\right)$ from $\boldsymbol{A}'$ to $\boldsymbol{B}'$. Let also π and ρ be the cofinal 1-cells from $\boldsymbol{A}$ to $\boldsymbol{A}'$ and from $\boldsymbol{B}$ to $\boldsymbol{B}'$, respectively, associated with the increasing sequence $(m_k)_{k\in\omega}$. Define

$$f^{(\infty)} = \mathrm{holim} f : \mathrm{holim}\boldsymbol{A} \to \mathrm{holim}\boldsymbol{B}$$
$$f'^{(\infty)} = \mathrm{holim} f' : \mathrm{holim}\boldsymbol{A}' \to \mathrm{holim}\boldsymbol{B}'$$
$$\pi^{(\infty)} = \mathrm{holim}\pi : \mathrm{holim}\boldsymbol{A} \to \mathrm{holim}\boldsymbol{A}'$$

and

$$\rho^{(\infty)} = \mathrm{holim}\rho : \mathrm{holim}\boldsymbol{B} \to \mathrm{holim}\boldsymbol{B}'.$$

Then there is a chain homotopy $\rho^{(\infty)} f^{(\infty)} \Rightarrow f'^{(\infty)}\pi^{(\infty)}$.

Proof. Define for $n \in \mathbb{Z}$, $H_n : C_n(\boldsymbol{A}) \to C_{n+1}(\boldsymbol{B}')$ by setting, for $z \in C_n(\boldsymbol{A})$ and $k \in \omega$,

$$H_n(z)_k = 0$$

and

$$H_n(z)_{k,k+1} = \sum_{m=m_k}^{m_{k+1}-1} f_{n+1}^{(m_k,m)}(z_{m,m+1})$$

We claim that H is a chain homotopy $\rho^{(\infty)} f^{(\infty)} \Rightarrow f'^{(\infty)}\pi^{(\infty)}$.

Suppose that $n \in \mathbb{Z}$, $k \in \omega$, and $z \in C_n(\boldsymbol{A})$. By definition, we have that

$$\left(f_n'^{(\infty)}\pi_n^{(\infty)} - \rho_n^{(\infty)} f_n^{(\infty)}\right)(z)_k = 0 = (d_{n+1}H_n + H_{n-1}d_n)(z)_k.$$

We also have that

$$\begin{aligned}
&\left(f_n'^{(\infty)}\pi_n^{(\infty)}\right)(z)_{k,k+1}\\
=\ & f_{n+1}^{(m_k)}\left(\pi_n^{(\infty)}(z)_{k,k+1}\right) + (-1)^n f_n^{(m_k,m_{k+1})}\left(\pi_n^{(\infty)}(z)_{k+1}\right)\\
=\ & f_{n+1}^{(m_k)}\left(z_{m_k,m_{k+1}}\right) + (-1)^n f_n^{(m_k,m_{k+1})}\left(z_{m_{k+1}}\right)\\
=\ & \sum_{m=m_k}^{m_{k+1}-1} f_{n+1}^{(m_k)}(p^{(m_k,m)}(z_{m,m+1})) + (-1)^n f_n^{(m_k,m_{k+1})}\left(z_{m_{k+1}}\right)
\end{aligned}$$

and

$$\begin{aligned}
&\left(\rho_n^{(\infty)} f_n^{(\infty)}\right)(z)_{k,k+1}\\
=\ & \varphi_n(z)_{m_k,m_{k+1}}\\
=\ & \sum_{m=m_k}^{m_{k+1}-1} p^{(m_k,m)}\left(f_n^{(\infty)}(z)_{m,m+1}\right)\\
=\ & \sum_{m=m_k}^{m_{k+1}-1} p^{(m_k,m)}\left(f_{n+1}^{(m)}(z_{m,m+1}) + (-1)^n f_n^{(m,m+1)}(z_{m+1})\right)\\
=\ & \sum_{m=m_k}^{m_{k+1}-1} p^{(m_k,m)} f_{n+1}^{(m)}(z_{m,m+1}) + (-1)^n \sum_{m=m_k}^{m_{k+1}-1} p^{(m_k,m)} f_n^{(m,m+1)}(z_{m+1})
\end{aligned}$$

thus

$$\begin{aligned}&\left(f_n^{\prime(\infty)}\pi_n^{(\infty)}-\rho_n^{(\infty)}f_n^{(\infty)}\right)(z)_{k,k+1}\\=&\sum_{m=m_k}^{m_{k+1}-1}\left(f_{n+1}^{(m_k)}p^{(m_k,m)}-p^{(m_k,m)}f_{n+1}^{(m)}\right)(z_{m,m+1})\\&+(-1)^n f_n^{(m_k,m_{k+1})}\left(z_{m_{k+1}}\right)+(-1)^{n+1}\sum_{m=m_k}^{m_{k+1}-1}p^{(m_k,m)}f_n^{(m,m+1)}(z_{m+1}).\end{aligned}$$

We now compute

$$\begin{aligned}(d_{n+1}H_n)(z)_{k,k+1}&=\partial\left(H_n(z)_{k,k+1}\right)+(-1)^{n+1}\left(p^{(k,k+1)}H_n(z)_{k+1}-H_n(z)_k\right)\\&=\partial\left(\sum_{m=m_k}^{m_{k+1}-1}f_{n+1}^{(m_k,m)}(z_{m,m+1})\right)=\sum_{m=m_k}^{m_{k+1}-1}\partial f_{n+1}^{(m_k,m)}(z_{m,m+1})\end{aligned}$$

and

$$\begin{aligned}(H_{n-1}d_n)(z)_{k,k+1}&=\sum_{m=m_k}^{m_{k+1}-1}f_n^{(m_k,m)}\left((d_nz)_{m,m+1}\right)\\&=\sum_{m=m_k}^{m_{k+1}-1}(f_n^{(m_k,m)}\partial)(z_{m,m+1})+(-1)^n\sum_{m=m_k}^{m_{k+1}-1}f_n^{(m_k,m)}\left(p^{(m,m+1)}(z_{m+1})-z_m\right)\end{aligned}$$

Therefore, we have that

$$\begin{aligned}&(d_{n+1}H_n+H_{n-1}d_n)(z)_{k,k+1}\\=&\sum_{m=m_k}^{m_{k+1}-1}\left(\partial f_{n+1}^{(m_k,m)}+f_n^{(m_k,m)}\partial\right)(z_{m,m+1})+(-1)^n\sum_{m=m_k}^{m_{k+1}-1}f_n^{(m_k,m)}\left(p^{(m,m+1)}(z_{m+1})-z_m\right)\end{aligned}$$

Recall that we have that

$$\partial f_{n+1}^{(m_k,m)}+f_n^{(m_k,m)}\partial=f_{n+1}^{(m_k)}p^{(m_k,m)}-p^{(m_k,m)}f_{n+1}^{(m)}$$

Therefore,

$$\sum_{m=m_k}^{m_{k+1}-1}\left(\partial f_{n+1}^{(m_k,m)}+f_n^{(m_k,m)}\partial\right)(z_{m,m+1})=\sum_{m=m_k}^{m_{k+1}-1}\left(f_{n+1}^{(m_k)}p^{(m_k,m)}-p^{(m_k,m)}f_{n+1}^{(m)}\right)(z_{m,m+1}).$$

It remains to prove that

$$\begin{aligned}&(-1)^n\sum_{m=m_k}^{m_{k+1}-1}f_n^{(m_k,m)}\left(p^{(m,m+1)}(z_{m+1})-z_m\right)\\=&(-1)^n f_n^{(m_k,m_{k+1})}\left(z_{m_{k+1}}\right)+(-1)^{n+1}\sum_{m=m_k}^{m_{k+1}-1}p^{(m_k,m)}f_n^{(m,m+1)}(z_{m+1})\end{aligned}$$

or, equivalently, that

$$\begin{aligned}&\sum_{m=m_k}^{m_{k+1}-1}f_n^{(m_k,m)}p^{(m,m+1)}(z_{m+1})+\sum_{m=m_k}^{m_{k+1}-1}p^{(m_k,m)}f_n^{(m,m+1)}(z_{m+1})\\=&f_n^{(m_k,m_{k+1})}\left(z_{m_{k+1}}\right)+\sum_{m=m_k}^{m_{k+1}-1}f_n^{(m_k,m)}(z_m).\end{aligned}$$

We have that

$$\begin{aligned}
&\sum_{i=m_k}^{m_{k+1}-1} f_n^{(m_k,i)} p^{(i,i+1)}\left(z_{i+1}\right) + \sum_{i=m_k}^{m_{k+1}-1} p^{(m_k,i)} f_n^{(i,i+1)}\left(z_{i+1}\right) \\
= &\sum_{i=m_k}^{m_{k+1}-1} \sum_{j=m_k}^{m-1} p^{(m_k,j)} f_n^{(j,j+1)} p^{(j+1,i+1)}\left(z_{i+1}\right) + \sum_{i=m_k}^{m_{k+1}-1} p^{(m_k,i)} f_n^{(i,i+1)}\left(z_{i+1}\right) \\
= &\sum_{i=m_k}^{m_{k+1}-1} \sum_{j=m_k}^{i} p^{(m_k,j)} f_n^{(j,j+1)} p^{(j+1,i+1)}\left(z_{i+1}\right) \\
= &\sum_{i=m_k+1}^{m_{k+1}} \sum_{j=m_k}^{i-1} p^{(m_k,j)} f_n^{(j,j+1)} p^{(j+1,i)}\left(z_{i}\right)
\end{aligned}$$

Similarly, we have that

$$\begin{aligned}
&f_n^{(m_k,m_{k+1})}\left(z_{m_{k+1}}\right) + \sum_{i=m_k}^{m_{k+1}-1} f_n^{(m_k,i)}\left(z_i\right) \\
= &\sum_{j=m_k}^{m_{k+1}-1} p^{(m_k,j)} f^{(j,j+1)} p^{(j+1,m_{k+1})}\left(z_{m_{k+1}}\right) + \sum_{i=m_k+1}^{m_{k+1}-1} \sum_{j=m_k}^{i-1} f_n^{(m_k,j)} f^{(j,j+1)} p^{(j+1,i)}\left(z_i\right) \\
= &\sum_{i=m_k+1}^{m_{k+1}} \sum_{j=m_k}^{i-1} p^{(m_k,j)} f_n^{(j,j+1)} p^{(j+1,i)}\left(z_i\right).
\end{aligned}$$

This concludes the proof. ☐

The following lemma establishing functoriality of the homotopy limit is the analogue in our context of [4, Lemma 18.7].

Lemma 4.9. Let $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ be towers of complexes. Suppose that $f = \left(m_k, f^{(k)}, f^{(k,k+1)}\right)$ is a 1-cell from $\boldsymbol{A}$ to $\boldsymbol{B}$ and $g = \left(k_t, g^{(t)}, g^{(t,t+1)}\right)$ is a 1-cell from $\boldsymbol{B}$ to $\boldsymbol{C}$. Define $h = gf$. Define also

$$f^{(\infty)} = \mathrm{holim} f$$
$$g^{(\infty)} = \mathrm{holim} g$$

and

$$h^{(\infty)} = \mathrm{holim}\left(gf\right).$$

Then there is a chain homotopy $g^{(\infty)} f^{(\infty)} \Rightarrow h^{(\infty)}$.

Proof. Let $\boldsymbol{A}'$, $\boldsymbol{A}''$, and $\boldsymbol{B}'$ be the towers of complexes $\left(A^{(m_k)}\right)_{k\in\omega}$, $(A^{(m_{k_t})})_{t\in\omega}$, and $\left(B^{(k_t)}\right)_{t\in\omega}$, respectively. Consider also:

- the level 1-cell $f' := \left(k, f^{(m_k)}, f^{(m_k,m_{k+1})}\right)$ from $\boldsymbol{A}'$ to $\boldsymbol{B}$;
- the level 1-cell $f'' := (t, f^{(m_{k_t})}, f^{(m_{k_t},m_{k_{t+1}})})$ from $\boldsymbol{A}''$ to $\boldsymbol{B}'$;
- the level 1-cell $\left(t, g^{(k_t)}, g^{(k_t,k_{t+1})}\right)$ from $\boldsymbol{B}'$ to $\boldsymbol{C}$.

Let $f'^{(\infty)}, f''^{(\infty)}, g'^{(\infty)}$ be the limit chain maps of f', f'', and g', respectively. Let π the cofinal 1-cell from $\boldsymbol{A}$ to $\boldsymbol{A}'$ corresponding to the sequence (m_k), π' be the cofinal 1-cell from $\boldsymbol{A}'$ to $\boldsymbol{A}''$ corresponding to the sequence (k_t), and ρ be the cofinal 1-cell from $\boldsymbol{B}$ to $\boldsymbol{B}'$ corresponding to the sequence (k_t). Let also $\pi^{(\infty)}$, $\pi'^{(\infty)}$, and $\rho^{(\infty)}$ be the corresponding limit chain maps. Then, by definition, we have that

$$f^{(\infty)} = f'^{(\infty)}\pi^{(\infty)} \quad \text{and} \quad g^{(\infty)} = g'^{(\infty)}\rho^{(\infty)} \quad \text{and} \quad h^{(\infty)} = g^{(\infty)} f''^{(\infty)}\pi'^{(\infty)}\pi^{(\infty)}.$$

By Lemma 4.8 we have a chain homotopy $\rho^{(\infty)} f'^{(\infty)} \Rightarrow f''^{(\infty)}\pi'^{(\infty)}$. Therefore, we have a chain homotopy

$$g^{(\infty)} f^{(\infty)} = g'^{(\infty)}\left(\rho^{(\infty)} f'^{(\infty)}\right)\pi^{(\infty)} \Rightarrow g^{(\infty)}\left(f''^{(\infty)}\pi'^{(\infty)}\right)\pi^{(\infty)} = h^{(\infty)}.$$

This concludes the proof. ☐

We can finally conclude the proof of Theorem 4.3.

Proof of Theorem 4.3. By Lemma 4.5, if $f : \boldsymbol{A} \to \boldsymbol{B}$ is a 1-cell, then $\mathrm{holim} f : \mathrm{holim} \boldsymbol{A} \to \mathrm{holim} \boldsymbol{B}$ is a chain map. Furthermore, the chain homotopy class $[\mathrm{holim} f] = \mathrm{holim}[f]$ of $\mathrm{holim} f$ only depends on the equivalence class of f up to 2-cells by Lemma 4.7. By Lemma 4.9 we have that $\mathrm{holim}[g \circ f] = \mathrm{holim}[g] \circ \mathrm{holim}[f]$. It is straightforward to verify that $\mathrm{holim}\,(1_{\boldsymbol{A}}) = 1_{\mathrm{holim} \boldsymbol{A}}$. This concludes the proof. □

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