Ultraproducts in logic and group theory

Martino Lupini

York University
Toronto, Canada

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Logic for metric structures

Logic for metric structures is a generalization of the usual first order logic.

It can be successfully used to **study algebraic, geometric or analytic structures** that

- either are naturally endowed with a (nontrivial) complete bounded metric (ex.: **valuation rings**);
- or can be described in terms of (almost) isometric embeddings into such structures (ex.: **sofic and hyperlinear groups**).

Languages

Definition

A **(metric) language** $\mathcal{L}$ is a collection of

- function symbols f,
- each one with prescribed arity $a(f) \in \mathbb{N}$,
- and prescribed continuity modulus $\omega(f) : \mathbb{R}_+ \to \mathbb{R}_+$.

An $\mathcal{L}$-**structure** $\mathcal{S}$ is given by

- a complete metric space S of diameter at most 1, called the support of $\mathcal{S}$;
- the interpretation in $\mathcal{S}$ of any function symbol f in $\mathcal{L}$, namely a uniformly continuous function $f^{\mathcal{S}} : S^n \to S$, where n is the arity of f, having $\omega(f)$ as uniform continuity modulus.

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For example, one can consider a language $\mathcal{L}_0$ containing

- a unique binary function symbol
- with the identity as uniform continuity modulus.

Structures

Definition

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For example, an $\mathcal{L}_0$-structure is

- a complete metric space with diameter at most 1,
- endowed with a binary operation that is 1-Lipschitz in every variable.

Metric symbol

The **special symbol** d will be considered a logic symbol.

The **interpretation** of d in a structure $\mathcal{S}$ will be the metric $d^{\mathcal{S}}$ of the metric space S.

In the following, all the metrics will be assumed to be complete and bounded by 1.

Language for bi-invariant metric groups

Suppose that G is a **group** endowed with a bi-invariant metric.

Both **multiplication** and **inversion** are isometric in every variable.

In particular, they are 1-Lipschitz in every variable.
One can then define the **language $\mathcal{L}_{Gr}$ of bi-invariant metric groups**, having as function symbols

- $\cdot$ of arity 2
- $(\quad)^{-1}$ of arity 1
- e of arity 0

each one with the **identity as uniform continuity modulus**.

A (particular) bi-invariant metric group can be seen as a structure in the language of bi-invariant metric groups.

Symmetric and unitary groups

The following bi-invariant metric groups have a key role in the study of sofic and hyperlinear groups.
For $n \in \mathbb{N}$, define

- $\mathfrak{S}_n$ the group of permutations of $\{1, 2, \ldots, n\}$
- U_n the group of $n \times n$ unitary matrices.

endowed with the bi-invariant metrics

- $d^{\mathfrak{S}_n}(\sigma, \tau) = \frac{1}{n} \left| \{ i \in \{1, 2, \ldots, n\} \mid \sigma(i) \neq \tau(i) \} \right|$
- $d^{U_n}(A, B) = \frac{1}{2\sqrt{n}} \|A - B\|_2$

Discrete groups as bi-invariant metric groups

A **discrete group** G can (and will be in the following) regarded as a bi-invariant metric group, endowed with the **trivial metric** d^G defined by

$$d^G(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{otherwise.} \end{cases}$$

Terms

If $\mathcal{L}$ is a language, the $\mathcal{L}$-**terms** are all the expressions constructed

- starting from some variable symbols $x, y, z \ldots$
- and composing them with the function symbols.

An $\mathcal{L}_{Gr}$-**term** (in the variables $x, y, z, \ldots$) is just a word (in $x, x^{-1}, y, y^{-1}, z, z^{-1}, \ldots$).

For example,

- $xyx^{-1}y^{-1}$
- e

are $\mathcal{L}_{Gr}$-terms, whose **interpretations** in a bi-invariant metric group are, respectively

- the function associating to two elements their commutator;
- the function constantly equal to the identity of the group.

Interpretation of terms

If $w = w(x_1, \ldots, x_n)$ is an $\mathcal{L}_{Gr}$-term (i.e. a word) in the variables $x_1, \ldots, x_n$,
its **interpretation**

$$w^G : G^n \to G$$

is the function associating to an n-tuple $(a_1, \ldots, a_n)$ of elements of G the element of G obtained replacing x_i with a_i in w and evaluating in G.
More generally, if $\mathcal{L}$ is any language, t is an $\mathcal{L}$-term and $\mathcal{S}$ is and $\mathcal{L}$-structure,
the **interpretation** $t^{\mathcal{S}}$ of t in $\mathcal{S}$ is a function on the support defined in terms of the interpretations in $\mathcal{S}$ of the function symbols of $\mathcal{L}$.

Basic formulas

A **basic $\mathcal{L}$-formula** is an expression of the form

$$d\left(t, \tilde{t}\right)$$

where $t, \tilde{t}$ are two $\mathcal{L}$-terms.

For example

$$d\left(xyx^{-1}y^{-1}, e\right)$$

is a basic $\mathcal{L}_{Gr}$-formula.

Its interpretation in a bi-invariant metric group is a function assigning to two elements the distance of their commutator from the identity.

Interpretation of basic formulas

If $w = w(x_1, \ldots, x_n)$ and $\tilde{w} = \tilde{w}(x_1, \ldots, x_n)$ are $\mathcal{L}_{Gr}$-terms and G is a bi-invariant metric group, the **interpretation** of $d(w, \tilde{w})$ in G is

$$d^G\left(w^G, \tilde{w}^G\right)$$

More generally, if $\mathcal{L}$ is a language, $t = t(x_1, \ldots, x_n)$, $\tilde{t} = \tilde{t}(x_1, \ldots, x_n)$ are $\mathcal{L}$-terms and $\mathcal{S}$ is an $\mathcal{L}$-structure, the **interpretation** of $d(t, \tilde{t})$ in $\mathcal{S}$ is

$$d^{\mathcal{S}}\left(t^{\mathcal{S}}, \tilde{t}^{\mathcal{S}}\right)$$

Formulas

We will call $\mathcal{L}$-**formulas** all the expressions constructed

- starting from basic $\mathcal{L}$-formulas,
- composing with continuous functions $q : [0,1]^n \to [0,1]$, and
- taking $\sup_x$ and $\inf_x$ with respect to some variable x.

For example

$$\sup_x \sup_y d\left(xyx^{-1}y^{-1}, e\right)$$

is an $\mathcal{L}_{Gr}$-formula (with no free variables).

Suppose that φ is the $\mathcal{L}_{Gr}$-formula

$$\sup_x \sup_y d\left(xyx^{-1}y^{-1}, e\right)$$

as before.

The interpretation φ^G of φ in a bi-invariant metric group G is the supremum of distances of commutators from the identity in G.

G will be abelian iff $\varphi^G = 0$.

The smaller φ^G is, the closer G is to be abelian.

Interpretations of more general formulas are defined in the expected way.

Unbounded metric structures

This model-theoretic framework can be generalized in order to deal with metric structures with not necessarily bounded metrics , such as

- **Banach spaces**;
- **C*-algebras**;
- **tracial von Neumann algebras**
- **II_1 factors**...

The general constructions and results that I will mention go through with in this broader setting.

Reduced products

Suppose that

- $(\mathcal{S}_n)_{n\in\mathbb{N}}$ is a **sequence of $\mathcal{L}$-structures**;
- $\mathcal{F}$ is a **filter** over $\mathbb{N}$.

Consider the pseudometric $d^{\mathcal{F}}$ on $\prod_{n\in\mathbb{N}} S_n$ defined by

$$d^{\mathcal{F}}\left((a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}\right) = \mathcal{F} - \limsup_{n\in\mathbb{N}} d^{\mathcal{S}_n}(a_n, b_n)$$

The **(metric) reduced product** $\mathcal{S}_{\mathcal{F}}$ of the sequence $(\mathcal{S}_n)_{n\in\mathbb{N}}$ with respect to $\mathcal{F}$ is the $\mathcal{L}$-structure that has

- as support, the metric space obtained as a quotient of the pseudometric space $\left(\prod_{n\in\mathbb{N}} S_n, d^{\mathcal{F}}\right)$;
- as interpretation $f^{\mathcal{S}_{\mathcal{F}}}\left(\left[(a_n)_{n\in\mathbb{N}}\right]\right) = \left[\left(f^{\mathcal{S}_n}(a_n)\right)_{n\in\mathbb{N}}\right]$ as interpretation of the function symbol f.

In particular,

- if $\mathcal{F}$ is the principal filter generated by $k \in \mathbb{N}$, $\mathcal{S}_{\mathcal{F}} \simeq \mathcal{S}_k$;
- if $\mathcal{F} = \mathcal{U}$ is a nonprincipal ultrafilter, then $\mathcal{S}_{\mathcal{U}}$ is an ultraproduct;
- if $\mathcal{F}$ is the Fréchet filter, then $\mathcal{S}_{\mathcal{F}}$ is the so called ℓ^∞/c_0-product.

Observe that, if $\mathcal{S}_n$ is separable for every $n \in \mathbb{N}$ then $\mathcal{S}_{\mathcal{F}}$ has density character at most $\mathfrak{c}$.

In particular, one can consider $\mathcal{L}_{Gr}$ and the sequence $(\mathfrak{S}_n)_{n\in\mathbb{N}}$.

The reduced product $\mathfrak{S}_{\mathcal{F}}$ has

- as support, $\prod_{n\in\mathbb{N}}\mathfrak{S}_n$ modulo the equivalence relation

$$(a_n)_{n\in\mathbb{N}} \sim (b_n)_{n\in\mathbb{N}} \text{ iff } \mathcal{F}-\limsup_{n\in\mathbb{N}} d^{\mathfrak{S}_n}(a_n, b_n) = 0,$$

  endowed with the metric

$$d^{\mathfrak{S}_{\mathcal{F}}}\left(\left[(a_n)_{n\in\mathbb{N}}\right], \left[(b_n)_{n\in\mathbb{N}}\right]\right) = \mathcal{F}-\limsup_{n\in\mathbb{N}} d^{\mathfrak{S}_n}(a_n, b_n);$$

- as operations, the **pointwise multiplication**

$$\left[(a_n)_{n\in\mathbb{N}}\right]\cdot\left[(b_n)_{n\in\mathbb{N}}\right] = \left[(a_n b_n)_{n\in\mathbb{N}}\right].$$

Sofic groups

Suppose G is a countable discrete group.

Definition

G is sofic if $\forall \varepsilon > 0$, $\forall F \subset G$ finite, there is $\varphi : G \to \mathfrak{S}_n$ (for some $n \in \mathbb{N}$) preserving all the operations and the distance up to ε on F.

This means that, for all $g, h \in F$,

- $d^{\mathfrak{S}_n}(\varphi(gh), \varphi(g)\varphi(h)) < \varepsilon$;
- $d^{\mathfrak{S}_n}\left(\varphi\left(g^{-1}\right), \varphi(g)^{-1}\right) < \varepsilon$;
- $d^{\mathfrak{S}_n}\left(\varphi\left(e^G\right), e^{\mathfrak{S}_n}\right) < \varepsilon$;
- $\left|d^{\mathfrak{S}_n}(\varphi(g), \varphi(h)) - d^G(g, h)\right| < \varepsilon$.

Universal sofic groups

If G is a countable discrete group, TFAE

- G is sofic;
- for every filter $\mathcal{F}$ over $\mathbb{N}$ extending the Fréchet filter, there is an isometric embedding ϕ :G$\rightarrow$ $\mathfrak{S}_{\mathcal{F}}$;
- for some filter $\mathcal{F}$ over $\mathbb{N}$ extending the Fréchet filter, there is an isometric embedding ϕ :G$\rightarrow$ $\mathfrak{S}_{\mathcal{F}}$.

An isometric embedding is a function **preserving all function symbols and the distance**.

The reduced products $\mathfrak{S}_{\mathcal{F}}$ are hence universal sofic groups: a countable group is sofic iff it is (isometrically) isomorphic to a subgroup of any of them.

The hyperlinear case

The same result holds if one replaces

- sofic groups with hyperlinear groups, and
- symmetric groups with unitary groups.

If G is a countable discrete group, TFAE

- G is hyperlinear;
- for every filter $\mathcal{F}$ over $\mathbb{N}$ extending the Fréchet filter, there is an isometric embedding ϕ :G$\rightarrow$ $U_{\mathcal{F}}$;
- for some filter $\mathcal{F}$ over $\mathbb{N}$ extending the Fréchet filter, there is an isometric embedding ϕ :G$\rightarrow$ $U_{\mathcal{F}}$.

Many outer automorphisms

Question

Are there outer automorphisms of $\mathfrak{S}_{\mathcal{U}}$?
How many of them?

Answer

If CH holds, yes.
There are indeed $2^{\aleph_1}$ outer automorphisms.

This can be proved using logic for metric structures, via the notion of countable saturation.

Countable saturation

Suppose that $\mathcal{S}$ is an $\mathcal{L}$-structure.

A sequence $(\varphi_n(x_1, \ldots, x_k))_{n \in \mathbb{N}}$ of $\mathcal{L}$-formulas with free variables $x_1, \ldots, x_k$ and possibly parameters from $\mathcal{S}$ is called

- satisfiable in $\mathcal{S}$ if there are $a_1, \ldots, a_k \in S$ such that $\varphi_n^{\mathcal{S}}(a_1, \ldots, a_k) = 0$ for every $n \in \mathbb{N}$;
- finitely satisfiable in $\mathcal{S}$ if every $n \in \mathbb{N}$, the finite sequence $(\varphi_i(x_1, \ldots, x_k))_{i \leq n}$ is satisfiable in $\mathcal{S}$.

We say that $\mathcal{S}$ is countably saturated if every **finitely satisfiable** in $\mathcal{S}$ sequence of formulas with parameters in $\mathcal{S}$ **is satisfiable** in $\mathcal{S}$.

Ultraproducts and countable saturation

Theorem

A ultraproduct over $\mathbb{N}$ of structures is countably saturated.

Theorem

If $\mathcal{S}$ is a countably saturated and has character density $\aleph_1$, then $|\mathrm{Aut}\,(\mathcal{S})| = 2^{\aleph_1}$.

Corollary

If CH holds, for every ultrafilter $\mathcal{U}$ over $\mathbb{N}$, $\mathfrak{S}_{\mathcal{U}}$ and $U_{\mathcal{U}}$ have $2^{\aleph_1}$ (isometric) outer automorphisms.

Only inner automorphisms?

Open problem

Is there, consistently, a $\mathcal{U}$ such that all the (isometric) automorphisms of $\mathfrak{S}_{\mathcal{U}}$ or $U_{\mathcal{U}}$ are inner?
And what about $\mathfrak{S}_{\mathcal{F}}$ for $\mathcal{F}$ a filter on $\mathbb{N}$ extending the Fréchet filter?

Theorem (Lücke and Thomas, 2010)

There is, consistently, a $\mathcal{U}$ such that the ultraproduct of the sequence of permutation groups **regarded as discrete groups** *has only inner automorphisms.*