Noncommutative analogs of the Gurarij Banach space

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The Gurarij Banach space

Definition

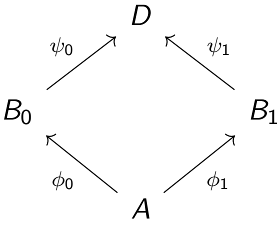
A separable Banach space $\mathbb{G}$ is Gurarij if for finite dimensional Banach spaces $E \subset F$, linear isometry $\phi : E \to \mathbb{G}$, and $\varepsilon > 0$, there is a linear map $\widehat{\phi} : F \to \mathbb{G}$ extending ϕ such that $\left\|\widehat{\phi}\right\| \left\|\widehat{\phi}^{-1}\right\| < 1 + \varepsilon$.

Gurarij (1966): Existence
Lusky (1976): Uniqueness
Gevorkjan (1974): Universality
Kubiś-Solecki (2011): Short proof of existence, uniqueness and universality
Ben Yaacov (2012): $\mathbb{G}$ as a Fraïssé limit (for metric structures)
Ben Yaacov (2013): $\mathrm{Iso}(\mathbb{G})$ is a universal Polish group (real scalars)
Ben Yaacov–Henson (2014): model-theoretic properties of $\mathbb{G}$ (real scalars)
Bartośova–López-Abad–Mbomb (2014): $\mathrm{Iso}(\mathbb{G})$ is extremely amenable

Fraïssé theory for metric structures

A class of finitely generated metric structures $\mathcal{C}$ is Fraïssé if

1. every finitely generated substructure of an element of $\mathcal{C}$ is in $\mathcal{C}$,
2. any two elements of $\mathcal{C}$ jointly embed into a third one, and
3. given $\phi_0 : A \to B_0$ and $\phi_1 : A \to B_1$ there are $D \in \mathcal{C}$ and $\psi_0 : B_0 \to D$ and $\psi_1 \to D$ such that the diagram below approximately commutes



4. the class $\mathcal{C}$ is separable.

Fraïssé theory for metric structures

Definition

A metric structure $\mathbb{S}$ is homogeneous if every isometric isomorphism between f.g. substructures is approximated arbitrarily well by an automorphism of $\mathbb{S}$.

Theorem (Ben Yaacov)

The map

$$\mathbb{S} \mapsto \mathrm{Age}(\mathbb{S}) = \{\textit{finitely generated substructures of } \mathbb{S}\}$$

is a 1:1 correspondence between separable homogeneous structures and Fraïssé classes.

Given a Fraïssé class $\mathcal{C}$ its limit is the unique separable homogeneous structure with age $\mathcal{C}$.

Some examples of Fraïssé classes and their limits

1. The class of finite metric, with limit the Urysohn space
2. The class of finite-dimensional Hilbert spaces, with limit ℓ_2
3. The class of finite probability algebras, with limit $\mathrm{MALG}(\lambda)$
4. The class of finite-dimensional Banach spaces, with limit $\mathbb{G}$

Some goals

- Find new examples of metric Fraïssé classes in functional analysis
- Use the knowledge about the limit to shed light on the model-theoretic properties of the class (joint with Isaac Goldbring)
- Find "easy to check" sufficient criteria to show that a class is Fraïssé
- Recognize as such already existing examples of Fraïssé limits in operator algebras (joint with Eagle-Farah-Hart-Kadets-Kalashnyk)
- Use the Fraïssé perspective to obtain information on the automorphism groups of the limits (joint with López-Abad)

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The quantum paradigm

The "quantization principle" in quantum mechanics consists in replacing functions with operator on the Hilbert space.

Let $B(H)$ be the space of bounded linear operators on the Hilbert space H.

Then $B(H)$ is a complex normed algebra with multiplicative identity 1 and involution $T \mapsto T^*$.

C*-algebras

A C*-algebra is a closed subalgebra $A \subset B(H)$ which is unital ($1 \in A$) and self-adjoint ($T^* \in A$ whenever $T \in A$).

The commutative C*-algebras are precisely those of the form $C(K)$ for some compact Hausdorff space K.

Mathematically, the quantization principle asserts that C*-algebras are the quantum (noncommutative) analog of compact Hausdorff spaces.

Quantum analog of Banach spaces

A Banach space is (concretely) a closed subspace of $C(K)$ for some compact Hausdorff space K.

Replacing $C(K)$ with (noncommutative) C*-algebras gives the quantum analog of Banach spaces.

Definition

An operator space is a closed linear subspace X of $B(H)$.

Definition

An operator system is an operator space $X \subset B(H)$ that

1. contains the identity operator 1 (the unit)
2. is closed under taking adjoints

Operator spaces

Suppose that $X \subset B(H)$ is an operator space.

The inclusions

$$M_n(X) \subset M_n(B(H)) \cong B\left(H^{\oplus n}\right)$$

induce matrix norms on X.

Operator spaces can be abstractly characterized among matricially normed spaces by Ruan's axiom

$$\left\| \sum_k \alpha_k^* . x_k . \beta_k \right\| \leq \left\| \sum_k \alpha_k^* \alpha_k \right\| \max_k \|x_k\| \left\| \sum_k \beta_k^* \beta_k \right\| .$$

One can similarly abstractly characterize operator systems in terms of the matrix norms and the unit (Blecher-Neal, Choi-Effros)

Completely bounded maps

A linear map $\phi : X \to Y$ between operator spaces is completely bounded

$$\|\phi\|_{cb} := \sup_n \left\| \phi^{(n)} \right\| < +\infty$$

where $\phi^{(n)}$ is the amplification

$$\begin{array}{rcl} M_n(X) & \to & M_n(Y) \\ [x_{ij}] & \mapsto & [\phi(x_{ij})] \end{array}$$

The notion of complete isometry and complete contraction are defined similarly.

Exact operator spaces

If E_0, E_1 are operator space of finite dimension n, their completely bounded distance

$$d_{cb}(E_0, E_1) = \inf \|\phi\|_{cb} \left\|\phi^{-1}\right\|_{cb}$$

when $\phi : E_0 \to E_1$ ranges among all linear isomorphisms.

Definition

An operator space X is exact if for every finite-dimensional $E \subset X$ and $\varepsilon > 0$ there exists $F \subset M_n$ such that $d_{cb}(E, F) < 1 + \varepsilon$.

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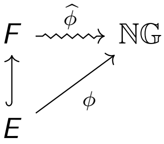
The Gurarij operator space

Definition (Oikhberg, 2006)

An exact operator space $\mathbb{NG}$ is (noncommutative) Gurarij if for finite dimensional exact operator spaces $E \subset F$, linear complete isometry $\phi : E \to \mathbb{NG}$, and $\varepsilon > 0$, there is a linear map $\widehat{\phi} : F \to \mathbb{G}$ extending ϕ such that $\left\|\widehat{\phi}\right\|_{cb} \left\|\widehat{\phi}^{-1}\right\|_{cb} < 1 + \varepsilon$.



Oikhberg showed that such a space exists.

Too many operator spaces

The restriction to exact operator spaces is natural due to the following:

Theorem (Junge-Pisier, 1995)

The space of 3-dimensional operator spaces with cb-distance has density character $2^{\aleph_0}$.

Corollary

There is no universal separable operator space.

The Gurarij operator space

Theorem

The Gurarij operator space $\mathbb{NG}$ is characterized by being the Fraïssé limit of the class of finite-dimensional exact operator spaces.

$\mathbb{NG}$ is unique, and universal for separable exact operator spaces

$\mathbb{NG}$ does not completely isometrically embed into an exact C-algebra*

Remark (Oikhberg)

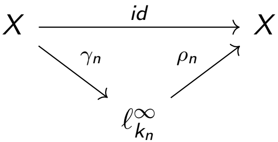
$\mathbb{NG}$ and $\mathbb{G}$ are not isomorphic as Banach spaces.

Lindenstrauss Banach spaces

Many known results about the Gurarij Banach spaces have analogs for $\mathbb{NG}$.

Definition

A separable Banach space X is Lindenstrauss if there is a sequence of contractions $\gamma_n : X \to \ell^\infty_{k_n}$ and $\rho_n : \ell^\infty_{k_n} \to X$ such that $\rho_n \circ \gamma_n \to id_X$.



Lazar-Lindenstrauss: X is Lindenstrauss iff it is L^1 predual.

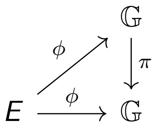
Lindenstrauss spaces and $\mathbb{G}$

Theorem (Wojtaszczyk, 1974)

Suppose that X is a separable Banach space. TFAE

1. *X is Lindenstrauss*
2. *X is isometric to a contractively complemented subspace of $\mathbb{G}$*

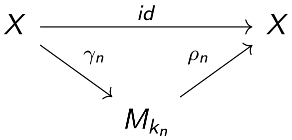
The second condition means that there exists a linear isometry $\phi : X \to \mathbb{G}$ and a contractive projection π of $\mathbb{G}$ onto the range of ϕ.



Nuclear operator spaces

Nuclear operator spaces are defined as Lindenstrauss Banach spaces

1. replacing ℓ_k^∞ with full matrix algebras M_k, and
2. replacing contractive linear maps with comp. cont. linear maps.



Remark

ℓ_k^∞ are precisely the finite-dimensional injective Banach spaces
$M_{k,d}$ and their ∞-sums are precisely the finite-dimensional operator spaces

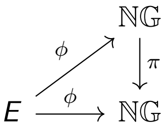
Nuclear operator spaces and $\mathbb{NG}$

Theorem

Suppose that X is a separable exact operator space. TFAE

1. *X is nuclear*
2. *X is completely isometric to a completely contractively complemented subspace of $\mathbb{NG}$*

The last condition means that there exists a linear complete isometry $\phi : X \to \mathbb{NG}$ and a completely contractive projection π of $\mathbb{NG}$ onto the range of ϕ.



Extreme amenability

Theorem (with Jordi López-Abad)

The group of surjective linear complete isometries of $\mathbb{NG}$ *is extremely amenable.*

The proof consists in establishing the approximate Ramsey property for approximately completely isometric linear maps between finite-dimensional exact operator spaces.

Operator systems

All we have seen so far goes through after

- replacing operator spaces with operator systems, and
- replacing linear maps with unital linear maps

Theorem

Finite-dimensional exact operator systems form a Fraïssé class with limit the Gurarij operator system $\mathbb{GS}$

The Gurarij operator system

Theorem (with Isaac Goldbring)

$\mathbb{GS}$ *is the unique existentially closed exact operator system*

The theory of $\mathbb{GS}$ *has* $2^{\aleph_0}$ *models (Junge-Pisier)*

$\mathbb{GS}$ *is the prime model of its theory, and the unique nuclear model*

The theory of $\mathbb{GS}$ *does not have quantifier elimination*

The class of operator systems does not have a model companion

Eagle-Farah-Kirchberg-Vignati (2015): analogous results for exact C*-algebras and $\mathcal{O}_2$

Some open problems

- Does $\mathbb{NG}$ have quantifier elimination?
- Is $\mathbb{NG}$ the prime model of its theory?
- Is the linear isometry group of the complex Gurarij space a universal Polish group?
- Give sufficient criteria for a Fraïssé class that ensure that the limit has universal automorphism group