First paper: Metrizable universal minimal flows of Polish groups have a comeagre orbit
Second paper: Polish groups with metrizable universal minimal flows

These two articles together solve a major open problem in abstract topological dynamics and the theory of minimal flows of Polish groups. The first paper (although second chronologically) shows that, if the universal minimal flow of a Polish group is metrizable, then it admits a comeager orbit. Using this result, the second paper shows that, whenever the universal minimal flow of a Polish group G is metrizable, then it can be obtained as the completion of the quotient of G by a co-precompact, extremely amenable, closed subgroup. In other words, this shows that the only known method to compute metrizable universal minimal flows is essentially the unique one.

Abstract topological dynamics studies, generally speaking, continuous actions of a Polish group G on compact Hausdorff spaces (*G-flows*). Since the class of G-flows is quite vast, it is natural to restrict one's attention to *minimal* G-flows. A G-flow is minimal if it admits no nontrivial closed G-invariant subset or, equivalently, if every orbit is dense. Minimal flows are in some sense the "irreducible elements" of the class of G-flows. A standard application of Zorn's lemma shows that any G-flow contains a minimal subflow. The natural notion of morphism between minimal G-flows is a factor map, which is a continuous G-equivariant map. Among all minimal G-flows there exists a canonical one, known as the universal minimal G-flow, and denoted by $M(G)$. The characterizing property of $M(G)$ is that it factors onto any other minimal G-flow. To see that such a minimal G-flow exists, one can take any minimal subflow of the product of *all* minimal G-flows. It is more subtle to see that such a minimal G-flow is indeed unique. This boils down to the fact, proved by Ellis, that the universal minimal flow $M(G)$ is *coalescent*, that is, any factor map from $M(G)$ to itself is necessarily injective.

These considerations naturally raise the problem of giving an explicit description of the universal minimal flow $M(G)$ for a given Polish group G. This is in general a difficult problem. For instance, whenever G is an infinite discrete group, $M(G)$ is isomorphic to a minimal left ideal of the Stone-Čech compactification βG of G (endowed with the canonical left translation action of G), and in particular not metrizable. More generally, $M(G)$ is not metrizable for every locally compact non compact second countable topological group G, as shown by Kechris, Pestov, and Todorocevic.

On the positive side, several examples have been found of naturally occurring Polish groups G whose universal minimal flow is not only metrizable, but even a single point. In this case, one says that the group G is *extremely amenable*. Examples of extremely amenable groups include the group of unitary operators on the separable infinite-dimensional Hilbert space (Gromov–Milman), and the group $\mathrm{Aut}(\mathbb{Q},<)$ of order-preserving bijections of the set of rational numbers (Pestov). Building on these examples, further explicit examples of metrizable universal minimal flows have been computed, including the universal minimal flows of the group of homeomorphisms of the circle (Pestov), and of the group S_∞ of permutations of the set of natural numbers (Glasner-Weiss).

The techniques developed in these examples inspired Kechris, Pestov, and Todorocevic to establish in 2005 in a very influential paper a general correspondence between Ramsey theory and extreme amenability. Kechris, Pestov, and Todorocevic focus on the case when G is given as the automorphism group of an ultrahomogeneous countably infinite first-order structure $\boldsymbol{F}$. This subsumes the cases of $\mathrm{Aut}(\mathbb{Q},<)$ and S_∞ mentioned above. In this setting, they showed that extreme amenability of G is equivalent to the assertion that the class of finite substructures of $\boldsymbol{F}$ (the *age* of $\boldsymbol{F}$) satisfies a Ramsey-theoretical statement that can be seen as an abstract "structural" version of the classical Ramsey theorem.

Building on this correspondence, Kechris, Pestov, and Todorcevic and, in a slightly more general form, Nguyen Van Thé provided a general method for computing the universal minimal flow of G in the case when the class $\boldsymbol{F}$ does not have the Ramsey property. Translated into topological terms, in a way that applies to an arbitrary Polish group G, this method consists in finding a "large" extremely amenable closed subgroup H of G. Here large means that H is *co-precompact*, namely the quotient space G/H (endowed with the uniformity induced by the right uniformity of G) is precompact, and its completion endowed with the canonical left-translation action of G is a minimal G-flow. All the known explicit computations of metrizable universal minimal flows follow this pattern. This naturally lead to the question of whether, whenever it is metrizable, the universal minimal flow of a Polish group is necessarily of this form. The non-Archimedean case having been settled by Zucker, the articles under review affirmatively answer this question for an arbitrary Polish group.

The first paper establishes a crucial first step in the proof, asserting that whenever $M(G)$ is metrizable, it contains a comeager orbit. This result is interesting on its own right, and it answers the "generic point problem" posed by

Angel, Kechris, and Lyons. The technique of the proof is simple and, at the same time, ingenious. The authors start from a canonical presentation of $M(G)$ as a subflow of the *Samuel compactification* $S(G)$ of G. This is a metric generalization of the Stone-Čech compactification of a discrete group, and can be briefly described as the Gelfand space of the commutative C*-algebra of bounded, right-uniformly continuous functions on G. Endowed with the canonical G-action, $S(G)$ is a G-flow, and any minimal subflow of $S(G)$ can identified with $M(G)$. When G is discrete, $S(G)$ is just the Stone-Čech compactification of G.

The authors provide a new insightful perspective on $S(G)$, by showing that several fundamental properties of the Stone-Čech compactification of a discrete group hold for $S(G)$ as well, as long as it is considered as a *topometric space*. A body of work due among others to Ben Yaacov, Berenstein, Ibarlucía, Melleray, and Tsankov has shown that topometric spaces provide the right framework to generalize results from discrete or non-Archimedan groups to arbitrary topological groups and, in the model-theoretic setting, from discrete structures to metric structures. A compact topometric space is a compact Hausdorff space endowed with a distinguished metric that induces a finer topology. There is a natural compatibility condition between the topology and the metric, requiring that the metric be lower semicontinuous. Any given compatible right-invariant metric d_R on G extends to a canonical metric on $S(G)$, which is the least metric on $S(G)$ with respect to which all the canonical extensions of bounded 1-Lipschitz maps on (X, d_R) are 1-Lipschitz. Such a metric turns $S(G)$ into a compact topometric space. In the case when G is discrete and $S(G) = \beta G$, the metric is just the trivial $\{0,1\}$-valued metric. The authors then obtained natural topometric versions of classical facts concerning the Stone-Čech compactification of a discrete group. Particularly, they show that in any metrizable subset of $S(G)$ the topology induced by the metric agrees with the subspace topology. This fact together with the existence of a G-equivariant retraction $S(G) \to M(G)$ allows them to verify a criterion due to Rosendal to establish the existence of a comeager orbit.

In the second paper, under the standing assumption that $M(G)$ is metrizable, and using the fact that in this case $M(G)$ contains a point x_0 with comeager orbit, the authors show that $M(G)$ is obtained as the completion of G/H for some closed co-precompact subgroup H. Furthermore, in such a case H is automatically extremely amenable, and it can be chosen to be the stabilizer of x_0. This is done in two steps, which are both nontrivial, and involve interesting new ideas. First one shows that the canonical left translation action of H on itself is *finitely oscillation stable*, which is a well known reformulation of extreme amenability. The second step consists of identifying $M(G)$ with the Samuel compactification $S(Y)$ of the homogeneous G-space $Y = G/H$. This is done by considering the canonical maps from $M(G)$ to $S(Y)$ and vice versa obtained from their respective universal properties, and showing that these maps are essentially the inverse of each other (modulo an automorphism). The assumption that x_0 has G_δ orbit is used together with Effros' theorem on G_δ orbits to conclude that the canonical inclusion from Y to $M(G)$ is a homeomorphism onto its image. The assumption that $M(G)$ is metrizable together with the isomorphism $M(G) \cong S(Y)$ then force $S(Y)$ to be equal to the completion of $Y = G/H$.

Several consequences concerning the structure of minimal G-flows are obtained from this result, under the standing assumption that $M(G)$ is metrizable. Particularly, the authors show that in this case, the classification problem for minimal G-flows up to G-isomorphism is *smooth* in the sense of Borel complexity theory. Furthermore, the authors show that *strong amenability* of G is equivalent to the assertion that $M(G)$ is a compact group (obtained as the quotient of G by an extremely amenable, *normal*, closed subgroup), and that this is always the case when G is a SIN group. Moreover, if G is CLI then every minimal G-flow is transitive. As yet another application, the universal minimal flows of the automorphism groups of several ultrahomogeneous countable graphs and digraphs are computed. These applications well illustrate how the main theorem of these papers provides a substantial amount of information on the structure of minimal flows. These results, together with the techniques developed in their proofs, with no doubt will be from now on part of the necessary toolkit for researchers working in the area of abstract topological dynamics and structural Ramsey theory.