The Lusky simplex

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Compact convex sets

A compact convex set is a w*-compact convex subset of a dual Banach spaces.

All compact convex sets are metrizable.

A compact convex set has a notion of:

- convexity,
- extreme point,
- extreme boundary,
- face.

Choquet simplices

Theorem (Choquet)

Any point is the barycenter of some boundary measure (representing measure).

A Choquet simplex is a compact convex set where every point has a unique representing measure.

The Poulsen simplex $\mathbb{P}$ is a canonical object in the theory of Choquet simplices.

The Poulsen simplex

Theorem (Poulsen 1961, Lindenstrauss–Olsen–Sternfeld 1978)

There exists a unique Choquet simplex $\mathbb{P}$ with dense extreme boundary.

- Universality: *Any Choquet simplex can be realized as a closed proper face of $\mathbb{P}$.*
- Homogeneity: *Any affine homeomorphism between closed proper faces of $\mathbb{P}$ extends to an automorphism of $\mathbb{P}$.*
- Uniqueness: *P is uniquely characterized among simplices by its universality and homogeneity properties*

Theorem (Bartošová–Lopez-Abad–L.–Mbombo, 2015)

The pointwise stabilizer of any proper closed face of $\mathbb{P}$ is extremely amenable.

Simplex spaces

A Lindenstrauss space is a Banach space whose dual is an L^1 space.

All the Banach spaces will be assumed to be separable

A simplex space is a pair (X, e_X) where X is a Lindenstrauss space $e_X \in \mathrm{Ball}(X)$ is an extreme point

If K is a simplex, define $A(K)$ to be the space of scalar-valued continuous affine functions on K and 1_K to be the function constantly equal to 1.

Simplices and simplex spaces

Theorem (Effros 1967, Hirsberg–Lazar 1983)

The following statement are equivalent:

1. K *is a simplex;*
2. $K = S(X, e_X)$ *for some simplex space* (X, e_X)

Furthermore $(X, e_X) \cong (A(K), 1_K)$

Continuous affine maps $\phi : K \to S$ are in 1:1 correspondence with unital linear maps $T : A(S) \to A(K)$

T is isometric iff ϕ is onto

The dual picture

Consider simplex spaces as Banach spaces with a distinguished element

Theorem (Conley–Törnquist 2013)

$A(\mathbb{P})$ is the Fraisse limit of the class of simplex spaces.

- **Universality** *Any simplex space unitally embeds as a subspace of $A(\mathbb{P})$ which is the range of a unital projection of norm 1*
- **Homogeneity** *Any two unital embeddings of a finite-dimensional simplex space into $A(\mathbb{P})$ are approximately conjugate.*
- **Uniqueness** *$A(\mathbb{P})$ is uniquely characterized among simplex spaces by the properties above*

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Lazar simplices

If X is a Banach space, then $\mathrm{Ball}(X^*)$ is a compact absolutely convex set.

Lazar characterized the absolutely convex sets K of the form $\mathrm{Ball}(X^*)$ for some real Lindenstrauss X (1971)

We will call these Lazar simplices

One can recover the Banach space X as the space $A_0(K)$ of scalar-valued absolutely convex continuous functions

Absolutely convex continuous functions $\phi : \mathrm{Ball}(X^*) \to \mathrm{Ball}(Y^*)$ are into 1:1 correspondence with nonexpansive linear maps $S : Y \to X$.

ϕ is onto iff S is isometric

Closed bifaces

The analog of a closed face in this context is a closed biface.

This notion was considered in 1971 by Effros and by Lazar–Lindenstrauss (closed facial section)

Definition

A closed biface of a Lazar simplex K is a closed absolutely convex $H \subset K$ of the form $\{0\}$ or $co(F \cup -F)$ for some face F of K.

The Lusky simplex

Theorem (Lusky 1976, 1979, L.)

There exists a unique Lazar simplex $\mathbb{L}$ (the Lusky simplex) with dense extreme boundary.

- **Universality** *Every Lazar simplex is a proper closed biface of $\mathbb{L}$*
- **Homogeneity** *Every absolutely affine homeomorphism between proper closed bifaces of $\mathbb{L}$ extends to an automorphism*
- **Uniqueness** *$\mathbb{L}$ is uniquely characterized among Lazar simplices by the homogeneity and universality properties above*

Theorem (Bartošová–Lopez-Abad–L.–Mbombo, 2015)

The pointwise stabilizer of any proper closed biface of $\mathbb{L}$ is extremely amenable.

The dual picture

Theorem (Lusky 1976, Wojtaszczyk 1972)

Let $\mathbb{G} = A_0(\mathbb{L})$

- **Universality** *Any Lindenstrauss space embeds as a subspace of* $\mathbb{G}$ *that is the range of a projection of norm* 1
- **Homogeneity** *Any two embeddings of a finite-dimensional Lindenstrauss space into* $\mathbb{G}$ *are approximately conjugate*
- **Uniqueness** $\mathbb{G}$ *is uniquely characterized by the universality and homogeneity properties above*

Thus $\mathbb{G}$ is the Gurarij Banach space first constructed by Gurarij (1966)

Analogous result for the complex scalars

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Ternary rings of operators

Now we work over the complex numbers

We denote by $\mathbb{G}$ the complex Gurarij space

Let H be a complex Hilbert space

$B(H)$ is the algebra of bounded linear operators on H, where product is composition of operators and $x \mapsto x^*$ is the adjoint map

Definition

A *ternary ring of operators* (TRO) is a closed subspace V of $B(H)$ that is invariant under the ternary product

$$(x, y, z) \mapsto xy^*z$$

A triple morphism between TROs is a linear map that preserves the triple product

Commutative ternary rings of operators

A TRO is commutative if it satisfies

$$xy^*z = zy^*x$$

Commutative TROs are called C_σ-spaces in the Banach space literature

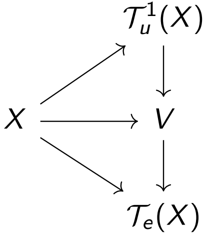
They are a subclass of Lindenstrauss spaces

Any Banach space X can be embedded in a commutative TRO V

We only consider such embeddings where V is generated by X as a TRO

The triple envelope and the universal commutative TRO

Among these embeddings, there exists two canonical ones



In particular for any Banach space X there exists a canonical triple morphism $\sigma_X : \mathcal{T}_u^1(X) \to \mathcal{T}_e(X)$

A new characterization of the Gurarij space

Theorem (L.)

The Gurarij space is the unique Lindenstrauss space such that σ_X is 1:1

Theorem (L.)

Let X be any Banach space. Define $X_0 = X$ and, recursively,

$$X_{n+1} = \mathcal{T}_u^1 (X_n) .$$

Then $\mathbb{G}$ is the direct limit of the sequence (X_n).

Universal automorphism group

Since $X \mapsto \mathcal{T}_u^1(X)$ is a functor we obtain:

Corollary

Every Banach space X admits a g-embedding into $\mathbb{G}$

$\mathrm{Aut}(\mathbb{G})$ *is a universal Polish group*

This was proved by Ben Yaacov (2014) over the reals

His argument does not apply to the complex case:
$\mathbb{R}$ is an injective metric space, but $\mathbb{C}$ is not

Work in progress: noncommutative analog of the above

The results mentioned today admit noncommutative analogs for

- the noncommutative Gurarij space (first constructed by Oikhbeg, 2006)
- the noncommutative Poulsen simplex (first constructed by Kirchberg–Wassermann, 1998)

Universality: OK
Extreme amenability: OK
Only missing: homogeneity

Need: noncommutative analogs of the Lazar and Lazar–Lindenstrauss selection theorems

Thank you!