Borel complexity and automorphisms of C*-algebras

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March 5th, 2014

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Invariant Complexity Theory

Invariant complexity theory studies the relative complexity of classification problems in mathematics.

In this framework a classification problem is seen as an equivalence relation on a standard Borel space.

This covers most naturally occurring classification problems.

Conjugacy of actions

Suppose for example that

- Γ is a countable discrete group,
- X is a compact Hausdroff space, and
- $\mathrm{Homeo}(X)$ is the group of auto-homeomorphisms of X.

An action of Γ on X is a homomorphism from Γ to $\mathrm{Homeo}(X)$.

Two actions α, β of Γ on X are conjugate if there is $\rho \in \mathrm{Homeo}(X)$ s.t.

$$\rho \circ \alpha(g) = \beta(g) \circ \rho$$

for every $g \in \Gamma$.

Conjugacy of actions

Endowed with the compact-open topology, $\mathrm{Homeo}(X)$ is Polish group.

The space $\mathrm{Homeo}(X)^{\mathbb{N}}$ of sequences of elements of $\mathrm{Homeo}(X)$ is Polish.

Fix a bijection between Γ and $\mathbb{N}$.

The space $\mathrm{Act}(\Gamma, X)$ of actions of Γ on X can be identified with a closed subspace of $\mathrm{Homeo}(X)^{\mathbb{N}}$.

The problem of classifying the action of Γ on X up to conjugacy can be seen as the problem of classifying the elements of $\mathrm{Act}(\Gamma, X)$ up to the corresponding equivalence relation on $\mathrm{Act}(\Gamma, X)$.

Reduction between equivalence relations

Let E and F be equivalence relations on standard Borel spaces X and Y.

A function f from X to Y such that for every $x, x' \in X$

$$f(x) F f(x') \quad \text{iff} \quad xEx'$$

is a reduction from E to F

A reduction assigns to the elements of X complete invariants from Y.

The following statements are equivalent:

- There is a reduction from E to F;
- F has at least as many classes as E.

Borel reducibility

Idea: Impose definability restrictions on the reduction.

This would give a nontrivial notion of comparison.

A natural definability assumption is requiring the reduction to be Borel.

Definition

E is Borel reducible to F if there is a Borel reduction from E to F. This is usually written $E \leq_B F$

The vast majority of concrete classification results in mathematics are witnessed by Borel reductions.

The spectral theorem

For example consider:

- a separable Hilbert space H, and
- the Polish group $U(H)$ of unitary operators on H.

Two unitary operators on H are conjugate if they are conjugate inside the group $U(H)$.

Two Borel measures on the circle are measure equivalent if they have the same null sets.

The following assertion is a consequence of the spectral theorem:
There is a Borel reduction

- from the relation of conjugacy of multiplicity-free unitary operators
- to the relation of measure equivalence of Borel measures on the circle.

Parametrization of countable discrete groups

Countable discrete groups are parametrized by functions from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$.

The set of such functions that code a group operation on $\mathbb{N}$ is Borel.

This can be seen as the standard Borel space of countable discrete groups.

Most classes of countable discrete groups form Borel subsets of this space:

- abelian groups,
- torsion free abelian groups,
- torsion free abelian groups of rank n,
- and many others...

Let

$$\cong_n$$

be the relation of isomorphism of torsion free abelian groups of rank n

It is not difficult to see that

$$\cong_n \leq_B \cong_{n+1}$$

for every $n \in \mathbb{N}$

Theorem (Thomas, 2003)

For every $n \in \mathbb{N}$

$$\cong_{n+1} \not\leq_B \cong_n$$

In other words the sequence

$$\cong_1 <_B \cong_2 <_B \cong_3 <_B \ \dots$$

is a strictly increasing chain of equivalence relations w.r.t. Borel reducibility

Smooth equivalence relations

Some equivalence relations can be used as a benchmark of complexity:

Let $=_X$ be the relation of identity of elements of a standard Borel space X.

An equivalence relation E is smooth if $E \leq_B =_X$ for some X

Smooth equivalence relations have the lowest complexity

Example

The relation of isomorphism of divisible abelian groups is smooth.

Example

Isomorphism of torsion free rank 1 abelian groups is not smooth.

Classifiable equivalence relations

Let $\mathcal{C}$ be a class of countable structures (such as groups, rings, etc.)

Denote by $\cong_{\mathcal{C}}$ the relation of isomorphisms of elements of $\mathcal{C}$

An equivalence relation E is classifiable (by countable structures) if

$$E \leq_B \cong_{\mathcal{C}}$$

for some class of countable structures $\mathcal{C}$

Example

Isomorphism of graphs has the maximal complexity within classifiable e.r.

Classification by countable structures

Many classification results involve countable structures as invariants.

Theorem (Elliott, 1977)

Approximately finite dimensional unital separable C-algebras are classified by their ordered K_0-group.*

Theorem (Kirchberg-Phillips, 1994)

Simple nuclear unital purely infinite separable C-algebras satisfying the Universal Coefficients Theorem are classified by their K_0 and K_1 groups.*

Fact (Farah-Toms-Tørnquist, 2012)

The K-theory of a C-algebra can be computed in a Borel way.*

Non-classifiability

The following equivalence relations are not classifiable:

- Conjugacy of multiplicity-free unitary operators on the Hilbert space (Kechris-Sofronidis, 2000);
- Isomorphism of simple nuclear unital C*-algebras (Farah-Toms-Tornquist, 2011);
- Unitary equivalence of irreducible representations of a non type I group (Hjorth, 1997);
- Unitary equivalence of irreducible representations of a non type I C*-algebra (Kerr-Li-Pichot, 2010);
- Unitary equivalence of automorphisms of a non type I C*-algebra (L., 2013).

The proofs of all these facts rely on Hjorth's theory of turbulence.

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Conjugacy of actions

Suppose that

- Γ is a countable discrete groups, and
- X is a compact metrizable space.

Define $\mathrm{Homeo}(X)$ to be the group of auto-homeomorphisms of X

An action $\Gamma \curvearrowright_{\phi} X$ is a group homomorphism $\phi : \Gamma \to \mathrm{Homeo}(X)$

Two actions ϕ, ϕ' are conjugate if there is $\alpha \in \mathrm{Homeo}(X)$ such that

$$\alpha \circ \phi(\gamma) \circ \alpha^{-1} = \phi'(\gamma)$$

for every $\gamma \in \Gamma$

Problem: Classify the actions of Γ on X up to conjugacy

$\mathbb{Z}$-actions

Consider the particular case $\Gamma = \mathbb{Z}$

$\mathbb{Z}$-actions on X correspond to single auto-homeomorphisms of X.

Conjugacy of actions correspond to conjugacy in $\mathrm{Homeo}(X)$

Problem: Classify elements of $\mathrm{Homeo}(X)$ up to conjugacy

Polish group of auto-homeomorphisms

$\mathrm{Homeo}(X)$ is a Polish group endowed with the compact-open topology

This is the topology of uniform convergence w.r.t. any compatible metric

We want to study the complexity of the relation

$$E_{\mathrm{Homeo}(X)}$$

of conjugacy inside $\mathrm{Homeo}(X)$

The zero-dimensional case

If X is zero-dimensional then

$$\mathrm{Homeo}(X) \cong \mathrm{Aut}(\mathcal{B}_X)$$

where $\mathcal{B}_X$ is the countable Boolean algebra of clopen subsets of X
Moreover

$$\mathrm{Aut}(\mathcal{B}_X) \subset S_\infty$$

It follows that

$$E_{\mathrm{Homeo}(X)} \leq_B E^X_{S_\infty}$$

for some $S_\infty \curvearrowright X$ and hence $E_{\mathrm{Homeo}(X)}$ is classifiable

The Cantor space

Theorem (Camerlo-Gao, 2001)

If $X = 2^\omega$ is the Cantor space, then

$$E_{\mathrm{Homeo}(2^\omega)}$$

is Borel complete

$E_{\mathrm{Homeo}(2^\omega)}$ has thus maximal complexity among classifiable e.r.

This is obtained of the corollary of

Theorem (Camerlo-Gao, 2001)

Isomorphism of countable Boolean algebras is Borel complete

Equivalently, by (a Borel version of) Stone's duality, homeomorphism of Boolean spaces is Borel complete

Interval and square

Consider now $\mathrm{Homeo}\left([0,1]\right)$ where $[0,1]$ is the unit interval

Theorem (Hjorth, 2000)

$E_{\mathrm{Homeo}([0,1])}$ *is classifiable*

The proof involves a suitable coding of auto-homeomorphisms of $[0,1]$ by countable structures

On the other hand increase of dimension entails an increase of complexity

Theorem (Hjorth, 2000)

$E_{\mathrm{Homeo}\left([0,1]^2\right)}$ *is not classifiable*

This is one of the earliest application of Hjorth's theory of turbulence

(Pre)turbulent Polish group action

Suppose that

- $G \curvearrowright X$ is a Polish group action
- V is an open neighborhood of 1 in G
- U is an open subset of X
- x is an element of U

The local orbit $\mathcal{O}(x, U, V)$ is the set of points of U that can be reached from x by applying elements of V without ever leaving U

The action $G \curvearrowright X$ is preturbulent if

1. every orbit is dense
2. every local orbit is somewhere dense

The action $G \curvearrowright X$ is turbulent if moreover it has meager orbits

Hjorth's turbulence theorem

Theorem (Hjorth, 2000)

If

- $G \curvearrowright X$ *is preturbulent, and*
- $S_\infty \curvearrowright Y$ *is an* S_∞*-action on a Polish space*

then E_G^X *is generically* $E_{S_\infty}^Y$*-ergodic:*

For any Borel map $f : X \to Y$ *sending* G*-orbits into* S_∞*-orbits,*
$\exists C$ *invariant comeager set such that* $f[C]$ *is contained in a single orbit.*

In particular if $G \curvearrowright X$ is turbulent then E_G^X is not classifiable

Consider

- the Polish group $H = \prod_{\mathbb{N}} \mathbb{Z}$ with the product topology
- the subgroup G of H defined by

$$(x_n) \in G \quad \text{iff} \quad \frac{x_n}{n} \to 0$$

- the action $G \curvearrowright H$ by translation

The action $G \curvearrowright H$ is turbulent.
Moreover any element h of H codes $\phi_h \in \mathrm{Aut}\left([0,1]^2\right)$ such that

$$Gh = Gh' \quad \text{iff} \quad \phi_h \text{ and } \phi_{h'} \text{ are conjugate.}$$

This shows that

$$E_G^H \leq_B E_{\mathrm{Homeo}([0,1]^2)}$$

In particular $E_{\mathrm{Homeo}([0,1]^2)}$ is not classifiable

C*-algebras

A unital C*-algebra A is a complex unital algebra with involution $x \mapsto x^*$ endowed with a norm making it a Banach space and satisfying

$$\|xy\| \leq \|x\| \, \|y\|$$

and

$$\|x^* x\| = \|x\|^2 .$$

All C*-algebras are henceforth assumed to be unital and separable

Suppose that X is a compact metrizable space

Let $C(X)$ be the space of complex-valued continuous functions on X

$C(X)$ is a C*-algebra with pointwise operations and sup norm

Abelian C*-algebras are exactly the ones of this form (Gelfand-Neimark)

Automorphisms groups

Suppose that A is a C*-algebra

Automorphisms of A are bijections $\alpha : A \to A$ preserving all the operations

Automorphisms of A automatically preserve the norm

The group of automorphisms of A is denoted by $\mathrm{Aut}(A)$

$\mathrm{Aut}(A)$ is a Polish group with the topology of pointwise norm convergence

Abelian case

In the abelian case $A = C(X)$ for some compact metrizable space X

There is an exact correspondence between auto-homeomorphisms of X and automorphisms of $C(X)$

Precisely

$$\begin{array}{rcl} \mathrm{Homeo}(X) & \to & \mathrm{Aut}\,(C(X)) \\ \phi & \mapsto & (f \mapsto f \circ \phi) \end{array}$$

is an isomorphism of Polish groups

In general $\mathrm{Aut}(A)$ can be thought as the group of auto-homeomorphisms of a noncommutative space

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Automorphisms of standard probability spaces

Suppose that (Y, ν) is a standard probability space

$\mathrm{Aut}(Y, \nu)$ is the group of measure preserving transformations of Y

The weak topology on $\mathrm{Aut}(Y, \nu)$ is the weakest topology such that

$$T \mapsto \nu\,(T[A] \triangle B)$$

measurable, for every $A, B \subset X$ Borel

$\mathrm{Aut}(Y, \nu)$ is a Polish group endowed with the weak topology

Problem: Study the Borel complexity of conjugacy in $\mathrm{Aut}(Y, \nu)$

The atomless standard probability space

Suppose that (X, μ) is the atomless standard probability space

$$A(X, \mu) = \{T \in \mathrm{Aut}(X, \mu) : T \text{ is aperiodic}\}$$

$T \in \mathrm{Aut}(X, \mu)$ is aperiodic if $T^n \neq 1$ for every $n \geq 1$

Lemma

$A(X, \mu)$ is a conjugation-invariant dense G_δ in $\mathrm{Aut}(X, \mu)$

Theorem (Foreman-Weiss 2004, Kechris 2010)

The action of $\mathrm{Aut}(X, \mu)$ on $A(X, \mu)$ is turbulent

Corollary

The elements of $\mathrm{Aut}(X, \mu)$ are not classifiable up to conjugacy.

Meager conjugacy classes

The shortest argument is due to Rosendal (2009)

Consider for V neighborhood of 1 in $\mathrm{Aut}\,(X,\mu)$ and $I \subset \mathbb{N}$ infinite

$$N(V,I) = \{T \in \mathrm{Aut}\,(X,\mu) : T^n \in V \text{ for some } n \in I\}$$

By Rohlin's lemma $N(V,I)$ is dense open in $\mathrm{Aut}\,(X,\mu)$.
It follows that

$$N\,(I) = \bigcap_V N(V,I)$$

is a conjugation-invariant dense G_δ subset of $\mathrm{Aut}(X,\mu)$

If C is a nonmeager orbit then $C \subset N\,(I)$ for every $I \subset \mathbb{N}$ infinite. Thus

$$C \subset \bigcap_{I \subset \mathbb{N} \text{ infinite}} N\,(I) = \{1\}$$

The Rosendal property

A similar argument works in general.

Suppose that:

- G is a Polish group;
- V is a neighborhood of 1 in G;
- $I \subset \mathbb{N}$ is infinite.

As before set

$$N(I, V) = \{g \in G : g^n \in V \text{ for some } n \in I\}$$

Definition

G has the Rosendal property if $N(I, V)$ is dense for every V and I

Lemma

If

- *$G \subset H$ are Polish groups,*
- *G has the Rosendal property,*

then the relation on G of conjugacy in H does not have a comeager class

Corollary

Under the assumptions of the theorem, if moreover the action of G on itself by conjugation is (generically) turbulent, then the elements of H are not classifiable up to conjugacy.

Not purely atomic standard probability space

Suppose that (Y, ν) is a not purely atomic standard probability space.

There is non-null $X \subset Y$ such that

$$(X, \nu_X)$$

is the atomless standard probability space, where

$$\nu(A) = \frac{1}{\nu(X)}\nu(A)$$

for $A \subset X$

Since

$$\mathrm{Aut}(X, \nu_X) \subset \mathrm{Aut}(Y, \nu)$$

it follows that conjugacy in $\mathrm{Aut}(Y, \nu)$ is not classifiable

L^∞ spaces

Define

$$L^\infty(Y,\nu) = \{f : Y \to \mathbb{C} \text{ Borel, bounded, up to mesure zero}\}$$

This is a C*-algebra with respect to pointwise operations, with norm

$$\|f\| = \inf\{\alpha \in \mathbb{R}_+ : f \leq \alpha \text{ a.e.}\}$$

The function

$$\begin{aligned} \tau : L^\infty(Y,\nu) &\to \mathbb{C} \\ f &\mapsto \int f d\nu \end{aligned}$$

defines the norm

$$\|f\|_\tau = \tau(f^*f)^{\frac{1}{2}} = \left(\int |f|^2 \, d\nu\right)^{\frac{1}{2}}$$

$(L^\infty(Y,\nu),\tau)$ is an example of tracial von Neumann algebra

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Traces

A trace τ on a C*-algebra A is a function

$$\tau : A \to \mathbb{C}$$

such that

1. τ is linear
2. $\tau(1) = 1$
3. $\tau(a^*a) \geq 0$
4. $\tau(ab) = \tau(ba)$

τ defines a the τ-norm

$$\|a\|_\tau = \tau(a^*a)^{\frac{1}{2}}$$

Tracial von Neumann algebras

A tracial von Neumann algebra (M, τ) is given by:

1. a C*-algebra M
2. a trace τ on M such that the unit ball of M is complete in τ-norm

The abelian tracial vN algebras are exactly the ones of the form $L^\infty(Y, \nu)$

Tracial von Neumann algebras will be assumed to be separable in τ-norm

Automorphism groups

Suppose that (M, τ) is a tracial von Neumann algebra

Denote by $\mathrm{Aut}(M, \tau)$ the group of automorphisms of M that preserve τ

$\mathrm{Aut}(M, \tau)$ with the topology of pointwise τ-norm convergence is Polish

$\mathrm{Aut}\left(L^\infty(Y, \nu), \tau\right) \cong \mathrm{Aut}(Y, \nu)$ (as Polish groups)

In general $\mathrm{Aut}(M, \tau)$ can be thought as the group of
measure preserving transformations of a noncommutative probability space.

Finite factors

Suppose that (M, τ) is a tracial von Neumann algebra

The center of M is

$$Z(M) = \{a \in M : ab = ba \text{ for all } b \in M\}$$

A tracial von Neumann algebra M is a finite factor if the center is trivial

Every tracial vN algebra is isomorphic to a direct integral of factors

The trace in a finite factor is unique

Any automorphism of a finite factor is automatically trace-preserving

II_1 factors

A matrix algebra $\mathbb{M}_n$ with the usual normalized trace is a finite factor

All finite dimensional finite factors are isomorphic to matrix algebras

Definition

A II_1 factor is an infinite dimensional finite factor

II_1 factors play a key role in the theory of von Neumann algebras

They were first studied in the 1930s by Murray and von Neumann

The hyperfinite II_1 factor

Consider the increasing sequence of matrix algebras

$$(\mathbb{M}_{2^n})_{n\in\omega}$$

regarded as finite factors , where

$$\begin{aligned} \mathbb{M}_{2^n} &\subset \mathbb{M}_{2^{n+1}} \\ A &= \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \end{aligned}$$

Define

$$\mathcal{R} = \overline{\bigcup_{n\in\omega} \mathbb{M}_{2^n}} \text{ (completion w.r.t. } \|\cdot\|_\tau \text{)}$$

This is the hyperfinite II_1 factor

It is the Fraïssé limit of the class of matrix algebras (as finite factors)

McDuff II_1 factors

The hyperfinite II_1 factor plays a key role in the study of II_1 factors.

It is the first example of II_1 factor considered by Murray and von Neumann

$\mathcal{R}$ is self-absorbing, i.e. $\mathcal{R} \otimes \mathcal{R} \cong \mathcal{R}$

If M is any II_1 factor, then $\mathcal{R} \subset M$

Definition

A II_1 factor M is McDuff if $M \otimes \mathcal{R} \cong M$

Being McDuff is equivalent to many other regularity properties

Goal: Show that the automorphisms of M are not classifiable

Inspired by the commutative case, let us consider first the case $M = \mathcal{R}$

Inner automorphisms

We need to identify the right analog of aperiodic transformation

Definition

An element u of $\mathcal{R}$ is unitary if $uu^* = u^*u = 1$

Any unitary element defines an automorphism of $\mathcal{R}$

$$\mathrm{Ad}(u) : a \mapsto uau^*$$

Automorphisms of this form are called inner

Remark: In the abelian case there are no nontrivial inner automorphisms

Lemma

Inner automorphisms are dense in $\mathrm{Aut}(\mathcal{R})$

Aperiodic automorphisms

Definition

An automorphism of $\mathcal{R}$ is aperiodic if none of its powers is inner

This is the analogue of aperiodic element of $\mathrm{Aut}\,(X, \mu)$

Denote by $A(\mathcal{R})$ the set of aperiodic elements of $\mathrm{Aut}(\mathcal{R})$

Lemma

$A(\mathcal{R})$ is a conjugation-invariant dense G_δ subset of $\mathrm{Aut}(\mathcal{R})$

Theorem (Kerr-Li-Pichot, 2010)

The action of $\mathrm{Aut}(\mathcal{R})$ on $A\,(\mathcal{R})$ is turbulent

Corollary

Automorphisms of $\mathcal{R}$ are not classifiable up to conjugacy

Nonclassification for McDuff factors

In order to show that the conjugacy classes in $\mathrm{Aut}(\mathcal{R})$ are meager, one in fact shows that $\mathrm{Aut}(\mathcal{R})$ has the Rosendal property

Therefore whenever G is a Polish group such that $G \supset \mathrm{Aut}(\mathcal{R})$, the relation of conjugacy in G is not classifiable

In particular if $M \cong M \otimes \mathcal{R}$ is McDuff then

$$\mathrm{Aut}(M) \cong \mathrm{Aut}\left(M \otimes \mathcal{R}\right) \supset \mathrm{Aut}\left(1 \otimes \mathcal{R}\right) \cong \mathrm{Aut}\left(\mathcal{R}\right)$$

Thus the automorphism of any McDuff II_1 factor are not classifiable

The same conclusion holds for the free group factors $L\left(\mathbb{F}_n\right)$

Open problems

Suppose that M is an arbitrary II_1 factor

Question

Are the automorphisms of M not classifiable up to conjugacy?

Theorem (Kerr-Li-Pichot, 2010)

The action of $\mathrm{Aut}(M)$ on itself by conjugation is generically preturbulent

To answer the question affirmatively, it would be enough to show that $\mathrm{Aut}(M)$ has meager conjugacy classes

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C*-algebras (joint work with David Kerr and Chris Phillips)

A is a C*-algebra, and $\mathrm{Aut}(A)$ is the Polish group of automorphisms of A

Problem: Determine the Borel complexity of conjugacy in $\mathrm{Aut}(A)$

The Jiang-Su algebra $\mathcal{Z}$ plays here an analog role as $\mathcal{R}$ for II_1 factors

$\mathcal{Z}$ was defined by Jiang and Su in 1999 in a groundbreaking paper

$\mathcal{Z}$ was constructed as direct limit of suitable "building blocks"

$\mathcal{Z}$ is in fact the Fraïssé limit of these building blocks

$\mathcal{Z}$ is self-absorbing, i.e. $\mathcal{Z} \otimes \mathcal{Z} \cong \mathcal{Z}$ (more in fact is true...)

$\mathcal{Z}$-absorbing C*-algebras

The analogue of McDuff factors are $\mathcal{Z}$-absorbing C*-algebras

A C*-algebra A is $\mathcal{Z}$-absorbing if $A \otimes \mathcal{Z} \cong A$

Conjecture (Toms-Winter)

If A is simple nuclear, the following statements are equivalent

1. *A is $\mathcal{Z}$-absorbing;*
2. *A has finite nuclear dimension;*
3. *A has strict comparison.*

Nuclear dimension is the "right" noncommutative generalization of the notion of dimension for compact metrizable spaces

The Toms-Winter conjecture has been verified in many interesting cases

Strongly aperiodic automorphisms

$\mathcal{Z}$ is endowed with a unique trace τ

The completion $\widehat{\mathcal{Z}}$ of $\mathcal{Z}$ with respect to the τ-norm is isomorphic to $\mathcal{R}$

Any $\alpha \in \mathrm{Aut}(\mathcal{Z})$ induces an automorphism $\widehat{\alpha}$ of $\widehat{\mathcal{Z}} \cong \mathcal{R}$

Definition

α is strongly aperiodic if $\widehat{\alpha} \in \mathrm{Aut}(\mathcal{R})$ is aperiodic

Define $A(\mathcal{Z})$ to be the set of strongly aperiodic automorphisms

Lemma (Sato)

$A(\mathcal{Z})$ is a conjugation-invariant dense G_δ subset of $\mathrm{Aut}(\mathcal{Z})$

Theorem

The action of $\mathrm{Aut}(\mathcal{Z})$ on $A(\mathcal{Z})$ is turbulent

Nonclassification

In order to show that $\mathrm{Aut}(\mathcal{Z})$ has meager conjugacy classes, one in fact shows that $\mathrm{Aut}(\mathcal{Z})$ is a Rosendal group

Therefore whenever G is a Polish group such that $G \supset \mathrm{Aut}(\mathcal{Z})$ the relation of conjugacy in G is not classifiable

In particular if A is a $\mathcal{Z}$-absorbing C*-algebra then

$$\mathrm{Aut}(A) \cong \mathrm{Aut}\,(A \otimes \mathcal{Z}) \supset \mathrm{Aut}\,(1 \otimes \mathcal{Z}) = \mathrm{Aut}(\mathcal{Z})$$

and the automorphisms of A are not classifiable up to conjugacy

The same conclusion holds when $A \cong A \otimes \mathbb{K}$
(consequence of a result of Kechris and Sofronidis about the unitary group)

Open problem

Suppose that A is an infinite-dimensional simple C*-algebra

Question

Is it true that the automorphisms of A are not classifiable up to conjugacy

The assumption that A is simple is necessary

There is a nonsimple A for which automorphisms are classifiable

An example is given by $A = C(2^{\omega})$, as we have seen

Outer conjugacy

Consider the relation E of cocycle conjugacy on $\mathrm{Aut}(\mathcal{Z})$ defined by

$$\alpha E \beta \quad \text{iff} \quad \mathrm{Ad}(u) \circ \gamma \circ \alpha \circ \gamma^{-1} = \beta$$

for some $\gamma \in \mathrm{Aut}(\mathcal{Z})$ and $u \in \mathcal{Z}$ unitary

Question

Is E classifiable by countable structures?

All strongly aperiodic automorphisms of $\mathcal{Z}$ are cocycle conjugate (Sato, 2009)

Since strongly aperiodic automorphisms form are a comeager set, the theory of turbulence seems not applicable to this situation

The Cuntz algebra

$\mathcal{O}_2$ is the universal C*-algebra generated by
2 partial isometries with orthogonal range projections summing up to 1

It was defined and studied by Cuntz in 1977 in a groundbreaking paper

It has a key role in the classification of simple nuclear C*-algebras

Theorem (Kirchberg, 1994)

$\mathcal{O}_2$ contain all exact C-algebras as subalgebras*

Theorem (Kirchberg, 1994)

$\mathcal{O}_2$ absorbs tensorially all simple nuclear unital C-algebras*

Some light on the Cuntz algebra

Theorem (Nakamura, 2000)

Aperiodic automorphisms of $\mathcal{O}_2$ are a comeager subset. Moreover they form a single cocycle conjugacy class.

Again, Hjorth's theory of turbulence does not seem to be applicable.

Theorem (Gardella-L., 2014)

Any classifiable Borel equivalence relation is Borel reducible to the relation of cocycle conjugacy of automorphisms of $\mathcal{O}_2$.

Theorem (Gardella-L., 2014)

The relation of cocycle conjugacy of automorphisms of $\mathcal{O}_2$ is **not Borel** *as a subset of* $\mathrm{Aut}\,(\mathcal{O}_2) \times \mathrm{Aut}\,(\mathcal{O}_2)$

The same holds even if one only considers automorphisms of order p

How the proof goes

Fix a prime number p.

With methods of Hjorth-Downey-Montalbán one can show the following:

Let $\mathcal{C}$ be the class of p-divisible torsion free abelian groups,
and $\cong_{\mathcal{C}}$ the relation of isomorphism of elements of $\mathcal{C}$.

Any classifiable Borel equivalence relation is Borel reducible to $\cong_{\mathcal{C}}$.

Moreover $\cong_{\mathcal{C}}$ is not Borel as a subset of $\mathcal{C} \times \mathcal{C}$.

Proposition (Gardella-L., 2014)

$\cong_{\mathcal{C}}$ is Borel reducible to the relation of cocycle conjugacy of automorphisms of $\mathcal{O}_2$ of order p.