The classification problem for finitely generated operator systems (joint work with Argerami, Coskey, Kalantar, Kennedy, Sabok)

Martino Lupini

York University

October 12th, 2014

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Positive cones

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An operator system $X \subset A$ inherits from A its order structure.

The inclusion $M_n(X) \subset M_n(A)$ defines a positive cone on all its matrix amplifications

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Such a map is completely positive if all its amplifications are positive

Choi-Effros' abstract characterization

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Theorem (Choi-Effros, 1977)

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satisfying suitable properties.

Operator systems and quantum information theory

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Classification of operator systems is understood up to complete order isomorphism.

Classification of operator systems

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The Connes embedding conjecture holds if and only if

$$\mathcal{S}_2 \otimes_{\mathrm{min}} \mathcal{S}_2 \cong \mathcal{S}_2 \otimes_{\mathrm{c}} \mathcal{S}_2$$

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Moreover: finitely generated o.s. are involved in the (re)formulation of many important properties of C*-algebras.

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Let us look a very nice case.

A very nice case

Suppose that $U, V \in B(H)$ are unitary operators

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Theorem (ACKKLS)

If $|\sigma(U)| \geq 5$ then TFAE

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Theorem (ACKKLS)

If $|\sigma(U)| \geq 5$ then TFAE

1. $\mathcal{OS}(U) \cong \mathcal{OS}(V)$
2. *there is a rigid motion of $\mathbb{T}$ mapping $\sigma(U)$ onto $\sigma(V)$*

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1. X is the homeomorphism group $\mathrm{Homeo}(K)$ of K with the Borel structure induced by the compact-open topology, and
2. E is the relation of conjugacy inside $\mathrm{Homeo}(K)$.

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(This can be proved in specific instances)

Borel reductions

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A Borel reduction correspond to an explicit way to assign to elements of X complete invariants up to E that are E'-equivalence classes

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Benchmarks of complexity

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Definition

An equivalence relation is smooth if it is Borel reducible to $=_{\mathbb{R}}$

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Borel complexity theory allows one to refine the analysis, and

observe that there are many different level of complexity beyond smooth

Some smooth examples

Example

The classification problem for divisible abelian groups is smooth

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A divisible abelian group is up to isomorphism

$$\bigoplus_{m_p} Z\left(p^{\infty}\right) \oplus \bigoplus_{n} \mathbb{Q}.$$

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Definition

E is classifiable by orbits if it is Borel reducible to the orbit equivalence relation associated with a continuous action of a Polish group on a Polish space

The classification problem for operator systems

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Answer (EFPRTT, Summer 2012, Banff)

Yes to both. In fact it follows from a general fact about the logic for metric structures.

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Later Argerami-Coskey-Kalantar-Kennedy-L.-Sabok considered the classification problem for finitely generated operator systems.

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History repeats itself...

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satisfying some properties.

Groups as structures

For example ordered group is a set endowed with

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$$\forall x, y, z \ \ (zxy = zy \Rightarrow \forall w \ \ wz = yx)$$

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A formula with no free variables can be interpreted in a given group, returning a truth value (true or false)

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A formula is then an expression involving function and relations, connectives, quantifiers, and d

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C*-algebras as metric structures

Example (Farah-Hart-Sherman)

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- functions for all the operations,
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A formula with no free variables can be interpreted in a structure, returning a real number

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- if (φ_n) is an enumeration of quantifier free formulas, then the sequence of the interpretations $\left(\varphi_n^X\right) \in \mathbb{R}^{\mathbb{N}}$ is an o.s. X is a complete invariant whose computation is given by a Borel map.

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In fact one can just consider formulas of the form

$$\sup_{x_1} \cdots \sup_{x_n} \varphi(x_1, \ldots, x_n)$$

where φ has no quantifiers

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Problem

Find sensible and easy to compute invariant(s) for finitely generated operator systems.

THANKS FOR YOUR ATTENTION!