Polish groupoids and the classification of operator algebraic varieties

Martino Lupini

California Institute of Technology

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Operator algebras

Let H be the separable complex Hilbert space.

Let $B(H)$ be the set of bounded linear operators $T : H \to H$.

$B(H)$ is a complex algebra with pointwise linear operations and multiplication=composition

It has a norm given by $\|T\| = \sup \{\|T\xi\| : \|\xi\| \leq 1\}$

Definition

An operator algebra is a closed subalgebra of $B(H)$.

Kernels

Suppose that $\mathbb{B}_d \subset \mathbb{C}^d$ is the Euclidean unit ball

For $d = \infty$ we convene that $\mathbb{C}^d = \ell^2$

A kernel on $\mathbb{B}^d$ is a positive definite analytic function $\mathbb{B}^d \times \mathbb{B}^d \to \mathbb{C}$

Any kernel K defines a Hilbert space $H(K)$ of analytic functions on K

$H(K)$ is the completion of the space of functions $k_\lambda = K(\cdot, \lambda)$ with respect to the inner product $\langle k_\lambda, k_\mu \rangle = K(\lambda, \mu)$

Multiplier algebras

The multiplier algebra $\mathrm{Mult}\,(H\,(K))$ of $H\,(K)$ is the set of analytic functions $\phi : \mathbb{B}_d \to \mathbb{C}$ such that, for every $f \in H\,(K)$, $\phi f \in H\,(K)$

This can be seen as an algebra of operators on $H\,(K)$ by identifying ϕ with the operator

$$\begin{array}{rcl} H\,(K) & \to & H\,(K) \\ f & \mapsto & \phi f \end{array}$$

The Hardy space

For example one can consider the kernel on $\mathbb{D} = \mathbb{B}_1$ defined by

$$K(\lambda, \mu) = \frac{1}{1 - \lambda\overline{\mu}}.$$

Then $H(K)$ is the Hardy space H^2: the space of analytic functions on $\mathbb{D}$ with square summable Taylor coefficients at zero

The corresponding multiplier algebra is the algebra $H^\infty(\mathbb{D})$ of bounded analytic functions on $\mathbb{D}$

The Drury-Arveson space

The complete Nevanlinna-Pick (NP) interpolation property is a regularity property for kernels coming from complex interpolation

Among the complete NP kernels, there exists a universal one: the Drury-Arveson kernel

This is an infinite-dimensional analg of the Hardy kernel on $\mathbb{B}_\infty \subset \ell^2$ defined by

$$K(\lambda, \mu) = \frac{1}{1 - \langle \lambda, \mu \rangle}$$

The associated Hilbert space is the Drury-Arveson space H^2_∞

Operator algebraic varieties

An operator algebraic variety is the set of common zeroes of a subset F of H^2_∞

Theorem (Agler-McCarthy, 2000)

Every kernel with the complete NP property is isomorphic to the restriction of the Drury-Arveson kernel to an operator algebraic variety

Corollary

The multiplier algebras $\mathrm{Mult}(H(K))$ *where* K *is a kernel with the complete NP property all have the form*

$$\mathcal{M}_V = \left\{ f|_V : f \in \mathrm{Mult}\left(H^2_\infty\right) \right\}$$

where $V \subset \mathbb{B}_\infty$ *is an operator algebraic variery*

Nonclassifiability of algebras

Theorem (Hartz–L, 2015)

The $\mathcal{M}_V$'s are not classifiable by countable structures up to isomorphism

A sequence (a_n) is admissible log-convex if $\frac{a_n}{a_{n+1}} \searrow 0$ and $\sum_n a_n = \infty$

Theorem (Hartz, 2015)

If (a_n) is admissible log-convex, then

$$K_{(a_n)}(\lambda, \mu) = \sum_n a_n \langle \lambda, \mu \rangle^n$$

is a complete NP kernel.
Furthermore

$$\mathrm{Mult}(H(K_{(a_n)})) \cong \mathrm{Mult}(H(K_{(a'_n)}))$$

if and only if (a_n) and (a'_n) have the same growth

Nonclassifiability of sequences

Theorem (Hartz–L, 2015)

Admissible log-convex sequences are not classifiable up to the relation of having the same growth

The proof of this result uses the theory of turbulence for Polish groupoids

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Groupoids

Definition

A groupoid G is a small category where every arrow is invertible

The set G^0 of objects can be see as a subset of the set G of arrows by identifying every object with its identity arrow

The orbit equivalence relation E_G is the relation of isomorphism in G

Definitions

A Polish groupoid is a groupoid endowed with a Polish topology that

- makes G^0 a G_δ subset of G
- makes composition and inversion of arrows continuous and open

Action groupoids

If $\Gamma \curvearrowright X$ is a Polish group action, one can consider the action groupoid

$$G = \Gamma \times X$$

with composition

$$(\gamma, \rho x)(\rho, x) = (\gamma\rho, x)$$

This groupoid completely encodes the action

More generally if $Y \subset X$ is a not necessarily invariant G_δ set, then $\Gamma \times Y \subset G$ is still a groupoid

Theorem (L., 2014)

Most results about the descriptive set theory of Polish group actions generalizes to Polish groupoids, including:

- *the Polishability theorem for Borel actions;*
- *the existence of universal actions;*
- *the characterization of Borel orbit equivalence relations.*

Turbulent groupoids

Suppose that G is a Polish groupoid, $x \in G^0$, and $x \in U \subset G$ is open

Definition

The local orbit $\mathcal{O}(x, U)$ is the set of objects that can be reached from x by applying elements of U

Definition

The object x is turbulent if every local orbit $\mathcal{O}(x, U)$ is somewhere dense

Definition

The groupoid G is generically turbulent if it has a turbulent object with dense orbit

The turbulence theorem

Theorem (Harz–L, 2015)

If G is generically turbulent, then the orbit equivalence relation E_G is generically S_∞-ergodic

Groupoids are the natural framework for the proof even in the case of actions, where one needs to restrict to a not necessarily invariant dense G_δ subset

Groupoid of sequences

Consider Polish group

$$\Gamma = \left\{ (g_n) \in \mathbb{R}^{\mathbb{N}} : \sum_n |g_n - 1| < +\infty \right\}$$

and the "partial action" of Γ on $(0,1)^{\mathbb{N}}$ by entrywise multiplication

This defines a turbulent Polish groupoid $G \subset \Gamma \times (0,1)^{\mathbb{N}}$

The corresponding orbit equivalence relation is given by $(s_n)\, E_G\, (s'_n)$ iff

$$\sum_n \left| \prod_{k \leq n} s_k - \prod_{k \leq n} s'_k \right| < +\infty$$

The reduction

The map

$$\begin{array}{rcl} (0,1)^{\mathbb{N}} & \to & (0,1)^{\mathbb{N}} \\ (s_n) & \mapsto & \left(\exp\left(-\sum_{k \leq n} \prod_{i \leq k} s_i \right) \right) \end{array}$$

is a generic homomorphism from E_G to the relation of having the same growth for admissible log-convex sequences

Furthermore any comeager subset of $(0,1)^{\mathbb{N}}$ contains two sequences whose images are do not have the same growth

A routine argument by contradiction shows that the relation of having the same growth for admissible log-convex sequences is not classifiable by countable structures

Questions

Problem

What about isometric isomorphisms of multiplier algebras?

Theorem

If one restricts to varieties in a fixed finite dimension then it has maximum complexity among essentially countable equivalence relations.