The noncommutative Poulsen simplex

Martino Lupini

California Institute of Technology

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Compact convex sets

Definition

A compact convex set is a compact convex subset of a locally convex topological vector space.

We will always assume compact convex sets to be also metrizable.

In a compact convex set one can define convex combinations.

This yields the notion of extreme point.

Definition

A point is extreme if it cannot be written as a convex combination of two other points.

Simplices

Definition

The extreme boundary $\partial_e K$ a compact convex set K is the set of its extreme points.

A boundary measure is a regular Borel measure supported on the boundary.

Theorem (Choquet, 1956)

Every point of a compact convex set is the barycenter of a boundary measure.

Definition

A compact convex set is called a (Choquet) simplex if such a measure is moreover unique.

For example the standard simplex $\Delta_n = \mathrm{Ball}\left(\ell_1^n\right)$ is a simplex with n extreme points.

The Poulsen simplex

In most "reasonable" simplices the extreme boundary is compact.

This can fail in a remarkable way:

Theorem (Poulsen, 1961)

There exists a simplex with dense extreme boundary.

Poulsen's construction is ad hoc and not canonical. However:

Theorem (Lindenstrauss–Olsen–Sternfeld, 1978)

There exists a unique simplex with dense extreme boundary.

Definition

The unique simplex with dense extreme boundary is the Poulsen simplex $\mathbb{P}$.

The Poulen simplex

The Poulsen simplex has remarkable universality and homogeneity properties.

Definition

A face of a simplex K is a subset $F \subset K$ such that no point in F is a convex combination of points outside F.

The extreme points are precisely the faces that contain a single point.

Denote by $\mathrm{Aut}(\mathbb{P})$ the group of affine homeomorphisms of $\mathbb{P}$.

Theorem (Lindenstrauss–Olsen–Sternfeld, 1978)

Every simplex K is affinely homeomorphic to a face of $\mathbb{P}$.
$\mathrm{Aut}(\mathbb{P})$ *acts transitively on the faces of $\mathbb{P}$ affinely homeomorphic to K.*

In particular $\mathrm{Aut}(\mathbb{P})$ acts transitively on its extreme points.

Function systems

Definition

A function system is a unital self-adjoint subspace $V \subset C(T)$ of a unital abelian C*-algebra $C(T)$.

We will assume all function systems to be separable.

If K is a compact convex set, then the space

$$A(K) = \{f : K \to \mathbb{C} : f \text{ is continuous and affine}\} \subset C(K)$$

is a function system with state space K.

Kadison's representation theorem

Theorem (Kadison, 1951)

The assignment

$$\begin{array}{rcl} \textit{compact convex sets} & \to & \textit{function systems} \\ K & \mapsto & A(K) \end{array}$$

is a contravariant equivalence of categories , with inverse

$$\begin{array}{rcl} \textit{function systems} & \to & \textit{compact convex sets} \\ V & \mapsto & S(V) \end{array}$$

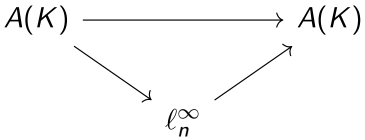
If Δ_n is the standard simplex, then $A(\Delta_n) = \ell_n^\infty$.

Simplices

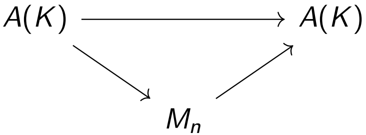
Suppose that K is a compact convex set.

The following statements are equivalent:

1. K is a simplex
2. $A(K)$ is Lindenstrauss



3. $A(K)$ is nuclear



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Noncommutative simplices

Definition

An operator system is a unital self-adjoint subspace $V \subset A$ of a unital C*-algebra A.

We will assume all operator systems to be separable.

Operator systems can be regarded as the noncommutative analog of compact convex sets.

Nuclear operator systems can be regarded as the noncommutative analog of simplices.

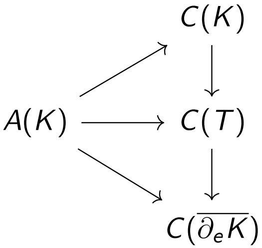
What should it be the analog of the Poulsen simplex?

Abelian C*-covers

Definition

An abelian C*-cover of a function system V is an inclusion $V \subset C(T)$ such that $C^*(V) = C(T)$.

Every function system has a maximal and a minimal abelian C*-cover



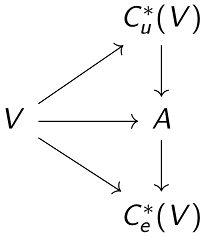
The canonical map $C(K) \to C(\overline{\partial_e K})$ is the restriction map.

C*-covers

Definition

A C*-cover of an operator system V is an inclusion $V \subset A$ such that $C^*(V) = A$.

Every operator system V has a maximal and a minimal C*-cover



Noncommutative Poulsen

A simplex K has dense extreme boundary iff the canonical map $C(K) \to C(\overline{\partial_e K})$ is 1:1.

Definition

A nuclear operator system V is noncommutative Poulsen if the canonical map $C_u^*(V) \to C_e^*(V)$ is 1:1

Theorem (Kirchberg–Wassermann, 1998)

There exists a nuclear operator system that is noncommutative Poulsen

Theorem (L, 2015)

There exists a unique noncommutative Poulsen simplex $\mathbb{GS}$ (the "Gurarij operator system")

Universality and homogeneity

$\mathbb{GS}$ satisfies noncommutative analogs of the universality and homogeneity properties of $\mathbb{P}$

Theorem (L, 2015)

If X is an operator system, then the following statements are equivalent

1. *X is exact*
2. $X \subset \mathbb{GS}$

The following statements are also equivalent

1. *X is nuclear*
2. *$X \subset \mathbb{GS}$ and there exist a ucp projection $\Pi_X : \mathbb{GS} \to X$*

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Flows

Suppose that G is a topological group.

A G-flow is a (not necessarily metrizable) compact space X endowed with a continuous action of G.

A factor Y of X is a G-flow with a continuous G-equivariant surjection $\pi : X \to Y$.

A G-flow is minimal if every orbit is dense.

A minimal G-flow is universal if it has any other minimal G-flow as a factor.

Universal minimal flows

Theorem (Ellis, 1960)

There exists a unique universal minimal G-flow $M(G)$.

$M(G)$ is not metrizable for any locally compact group G

$M(G)$ is “small” for “large” groups.

Definition

G is extremely amenable if $M(G)$ is a singleton

This is equivalent to the assertion that any G-flow has a fixed point

Theorem (Bartosova–Lopez-Abad–Lupini–Mbombo, 2015)

If F is a proper face of $\mathbb{P}$, then the pointwise stabilizer $\mathrm{Aut}_F(\mathbb{P})$ is extremely amenable

Theorem (Bartosova–Lopez-Abad–Lupini–Mbombo, 2015)

$M(\mathrm{Aut}(\mathbb{P}))$ is $\mathbb{P}$ endowed with the canonical action of $\mathrm{Aut}(\mathbb{P})$

This can be reformulated in terms of a Ramsey-theoretical statement about finite-dimensional function system.
This statement is proved by applying the Dual Ramsey Theorem of Graham and Rotschild

The noncommutative Poulsen simplex and dynamics

Theorem (Bartosova–Lopez-Abad–Lupini–Mbombo, 2015)

If $X \subset \mathbb{GS}$ is a nuclear operator system and $\Pi_X : \mathbb{GS} \to X$ is the canonical ucp projection, then the stabilizer

$$\mathrm{Aut}_X(\mathbb{GS}) = \{\alpha \in \mathrm{Aut}(\mathbb{GS}) : \Pi_X \circ \alpha = \Pi_X\}$$

is extremely amenable

Theorem (Bartosova–Lopez-Abad–Lupini–Mbombo, 2015)

$M(\mathrm{Aut}(\mathbb{GS}))$ is the state space $S(\mathbb{GS})$ endowed with the canonical action of $\mathrm{Aut}(\mathbb{GS})$

This can be reformulated in terms of a Ramsey-theoretical statement about finite-dimensional operator systems.

Noncommutative flows

The examples of $\mathbb{P}$ and $\mathbb{GS}$ suggest that the noncommutative universal minimal flow of $\mathrm{Aut}\,(\mathbb{GS})$ should be $C_e^*\,(\mathbb{GS}) = C_u^*\,(\mathbb{GS})$

The theory of noncommutative universal minimal flows has not been developed

Definition

A noncommutative G-flow is a unital C*-algebra A endowed with an action of G

Noncommutative flows

Definition

A noncommutative G-flow is minimal if it has no nontrivial invariant hereditary subalgebras

Definition

A noncommutative G-flow is universal if it contains any minimal noncommutative G-flow

Problem

Does it exist a universal minimal noncommutative G-flow? Is it unique?