Classification of automorphisms of C*-algebras

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Classification problems

Suppose that:

- X is a class of **objects**;
- E is an **equivalence relation** on X.

Classifying the objects of E up to X usually means assigning (in a "constructive" way) to every object X some invariant, in such a way that two objects are related by E if and only if the associated invariants are equal (or equivalent in some sense).

E.g., the class of **AF algebras** is classified up to $*$-**isomorphism** by the K_0 group, since

- $K_0(A)$ is constructively associated to A;
- two AF algebras A and B are isomorphic iff $K_0(A)$ and $K_0(B)$ are isomorphic as dimension groups (Elliott, 1976).

Descriptive set-theoretic framework

Descriptive set theory offers a framework where one can study and compare the **complexity of different classification problems**.

In most concrete cases of classification problems,

- the class X of objects is naturally endowed with a Polish topology,
- the relation E is a **Borel** (or at least analytic) subset of $X \times X$ endowed with the product topology.

For example, suppose that

- X is the set $\mathrm{Aut}\,(A)$ of **automorphisms of a C*-algebra** A;
- E is the relation of **unitary equivalence**.

$\mathrm{Aut}\,(A)$ is a Polish space with respect to the **point-norm topology** , and E is a Borel equivalence relation on $\mathrm{Aut}\,(A)$.

Borel reduction

One now wants to study and compare pairs (X, E), where

- X is a Polish space;
- E is an analytic equivalence relation.

A **notion of comparison** is given by (Borel) reduction.

Definition

A (Borel) reduction from (X, E) to (X', E') is a function $f : X \to X'$ s.t.

- f is Borel;
- for $x, y \in X$, xEy iff $f(x)\, E' f(y)$.

If there is a (Borel) reduction from (X, E) to (X', E'),
we say that E is **(Borel) reducible** to E', and we write

$$E \leq E'.$$

Benchmark equivalence relations

These are benchmark relations to be used as **measures of complexity:**

1. the relation $=_{\mathbb{R}}$ of **equality of real numers**;
2. the **isomorphism** relation $\simeq_{\mathcal{C}}$ in a class $\mathcal{C}$ of **countable structures** (such as groups, rings etc...).

These (families of) relations give a hierarchy of complexity, since

$$=_{\mathbb{R}} \leq \simeq_{\mathcal{C}}$$

Definition

An equivalence relation E is called

1. **smooth** if it is reducible to $=_{\mathbb{R}}$;
2. **classifiable by countable structures** if it is reducible to $\simeq_{\mathcal{C}}$ for some class $\mathcal{C}$ of countable structures.

Glimm's theorem

Suppose that

- A is a separable C*-algebra;
- $\mathcal{H}$ is a separable Hilbert space;
- $\mathrm{Irr}_{\mathcal{H}}(A)$ is the Polish space of irreducible representations of A on $\mathcal{H}$, endowed with the **topology of pointwise norm convergence**;
- $\sim^{A}_{u.e.}$ is the relation of unitary equivalence on $\mathrm{Irr}_{\mathcal{H}}(A)$, namely

$$\pi \sim^{A}_{u.e.} \pi' \qquad \text{iff} \qquad \pi' = Ad(u) \circ \pi \text{ for some } u \in U(\mathcal{H})$$

Theorem (Glimm, 1961)

For a separable C-algebra A, TFAE*

1. *A is* **type I**
2. *the relation* $\sim^{A}_{u.e.}$ *on* $\mathrm{Irr}_{\mathcal{H}}(A)$ *is* **smooth**

Non-classification by countable structures

Glimm's result was refined by Kerr-Li-Pichot and, independently, Farah.

Theorem (2009)

If A is a separable **non type I** *C^*-algebra, then the relation $\sim_{u.e.}$ is* **not classifiable by countable structures**.

The relation $E^{\ell_2}_{\mathbb{R}^{\mathbb{N}}}$ of ℓ_2-equivalence in $\mathbb{R}^{\mathbb{N}}$ is not classifiable.
If $\mathbb{M}_{2^\infty}$ is the CAR algebra and A is any non type I C*-algebra, then

$$E^{\ell_2}_{\mathbb{R}^{\mathbb{N}}} \leq \sim^{\mathbb{M}_{2^\infty}}_{u.e.} \leq \sim^{A}_{u.e.}$$

The first reduction is given by

$$\begin{aligned} \mathbb{R}^{\mathbb{N}} &\to \mathcal{P}\left(\mathbb{M}_{2^\infty}\right) \\ (t_n)_{n\in\mathbb{N}} &\mapsto \bigotimes_{n\in\mathbb{N}} \omega_{t_n}, \end{aligned}$$

where ω_{t_i} is the vector state on $\mathbb{M}_2$ associated to the vector $(cos\, t_n, sin\, t_n)$

Phillips' theorem

Suppose that

- A is a separable unital C*-algebra;
- $U(A)$ is the unitary group of A;
- $\mathrm{Aut}(A)$ is the automorphism group of A endowed with the **point-norm topology**;
- $\approx^{A}_{u.e.}$ is the relation of unitary equivalence on $\mathrm{Aut}(A)$, namely

$$\alpha \approx^{A}_{u.e.} \beta \qquad \text{iff} \qquad \alpha = \mathrm{Ad}(u) \circ \beta \text{ for some } u \in U(A)$$

Theorem (J. Phillips, 1985)

If A is a separable unital C-algebra, TFAE*

1. *A has **continuous trace**;*
2. *$\approx^{A}_{u.e.}$ is **smooth**.*

Non-classification by countable structures

This result can be refined for the class of all $\mathcal{Z}$-stable unital C*-algebras.

Theorem (L., 2012)

If A is a $\mathcal{Z}$-stable unital C-algebra, then the relation $\approx^{A}_{u.e.}$ is* **not classifiable by countable structures**.

Fix an element $a \in \mathcal{Z}_{sa}$ such that $\exp(ia)$ is not in the center.
Define the continuous homomorphism

$$\begin{array}{cccc} \Psi : & \mathbb{R}^{\mathbb{N}} & \to & \mathrm{Aut}\left(A \otimes \mathcal{Z}^{\otimes \mathbb{N}}\right) \\ & (t_n)_{n\in\mathbb{N}} & \mapsto & id_A \otimes \bigotimes_{n\in\mathbb{N}} \mathrm{Ad}\left(\exp\left(it_n a\right)\right) \end{array}$$

Observe that

$$\Psi\left[\ell^2\right] \subset \mathrm{Inn}\left(A \otimes \mathcal{Z}^{\otimes \mathbb{N}}\right)$$

and, if $\mathbf{1}$ is the sequence constantly equal to 1, $\Psi(\mathbf{1}) \notin \mathrm{Inn}\left(A \otimes \mathcal{Z}^{\mathbb{N}}\right)$.

Lemma (Hjorth, 2000)

If $\mathcal{C}$ is a class of countable structures and $F : \mathbb{R}^{\mathbb{N}} \to \mathcal{C}$ is a Borel function such that

$$F(x+z) \simeq_{\mathcal{C}} F(z) \quad \text{for all } x \in \mathbb{R}^{\mathbb{N}} \text{ and } z \in \ell_2,$$

then there is a comeager subset X of $\mathbb{R}^{\mathbb{N}}$ such that

$$F(x) \simeq_{\mathcal{C}} F(x') \quad \text{for all } x, x' \in X.$$

Suppose by contradiction that $\approx_{u.e.}^{A\otimes\mathcal{Z}^{\otimes\mathbb{N}}}$ is classifiable by countable structures.

Then, there are

- a class $\mathcal{C}$ of countable structures, and
- a Borel reduction Φ from $\approx_{u.e.}^{A\otimes\mathcal{Z}^{\otimes\mathbb{N}}}$ to $\simeq_{\mathcal{C}}$.

Since Ψ is a group homomorphism and $\Psi[\ell_2] \subset \mathrm{Inn}\left(A\otimes\mathcal{Z}^{\otimes\mathbb{N}}\right)$, the composition $\Phi\circ\Psi : \mathbb{R}^{\mathbb{N}} \to \mathcal{C}$ satisfies the hypothesis of the lemma.

Let $X \subset \mathbb{R}^{\mathbb{N}}$ be comeager such that, for $x, x' \in X$,

$$\Phi \circ \Psi(x) \simeq \Phi \circ \Psi(x')$$

and, hence

$$\Psi(x) \approx_{u.e.}^{A \otimes \mathcal{Z}^{\otimes \mathbb{N}}} \Psi(x') .$$

This implies that $\Psi[X]$ is contained in a coset of $\mathrm{Inn}\left(A \otimes \mathcal{Z}^{\otimes \mathbb{N}}\right)$.
Thus, X is contained in a coset of $H = \Psi^{-1}\left[\mathrm{Inn}\left(A \otimes \mathcal{Z}^{\otimes \mathbb{N}}\right)\right]$.
Since

- $\Psi\left[\ell^2\right] \subset \mathrm{Inn}\left(A \otimes \mathcal{Z}^{\otimes \mathbb{N}}\right)$, and
- $\Psi(\mathbf{1}) \notin \mathrm{Inn}\left(A \otimes \mathcal{Z}^{\otimes \mathbb{N}}\right)$

one has

$$\ell_2 \subset H \subsetneq \mathbb{R}^{\mathbb{N}}$$

and hence H is a proper dense Borel subgroup of $\mathbb{R}^{\mathbb{N}}$.
It follows that H and all the cosets of H are meager,
contradicting the fact that X is comeager and contained in a coset of H.

The proof shows in fact that, if A is unital, separable and $\mathcal{Z}$-**stable**, then even approximately inner centrally trivial automorphisms of A are **not classifiable up to unitary equivalence by countable structures**.

Conjecture

If A is a unital separable C-algebra wich does not have continuous trace, then $\approx^{A}_{u.e.}$ is* **not classifiable by countable structures***.*