The complexity of the relation of unitary equivalence of automorphisms of separable C*-algebras

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If A is a separable C*-algebra, denote by $\operatorname{Aut}(A)$ the Polish group of automorphisms of A endowed with the topology of pointwise convergence, and $\operatorname{Inn}(A)$ the Borel subgroup of inner automorphisms. Two automorphisms α, β of A are said to be unitarily equivalent if $\alpha \circ \beta^{-1} \in \operatorname{Inn}(A)$. This defines a Borel equivalence relation $E_A^{u.e.}$ on $\operatorname{Aut}(A)$. The main result presented here concerns the Borel complexity of the equivalence relation $E_A^{u.e.}$.

The study of Borel complexity of Borel (or analytic) equivalence relations on standard Borel spaces is one of the main applications of descriptive set theory (see [2] for an introduction to this subject). If E and E' are two analytic equivalence relations on standard Borel spaces X and X' respectively, E is said to be Borel reducible to E' if there is a Borel map $f : X \to X'$ such that, for every $x, y \in X$, xEy if and only if $f(x) E' f(y)$. This offers a notion of comparison that allows one to confront the complexity of different equivalence relations. Some distinguished equivalence relations can be used as benchmark of complexity. Among these, the relation $=_{\mathbb{R}}$ of equality real numbers and the relation $\simeq_{\mathcal{C}}$ of isomorphism within some class of countable structures $\mathcal{C}$. An analytic equivalence relation E is called smooth if it is Borel reducible to $=_{\mathbb{R}}$, and classifiable by countable structures if it is Borel reducible to $\simeq_{\mathcal{C}}$ for some class $\mathcal{C}$ of countable structures. Since $=_{\mathbb{R}}$ is Borel reducible to $\simeq_{\mathcal{C}}$ for any class $\mathcal{C}$ of countable structures with uncountably many isomorphism classes, a smooth equivalence relations is, in particular, classifiable by countable structrues. Smooth equivalence relations are rare and have the lowest Borel complexity. An example of smooth equivalence relation is the relation of unitary equivalence of irreducible representation of a given separable type I C*-algebra. Much wider is the class of equivalence relations that are classifiable by countable structures. This can be regarded as the class of equivalence relations for which one can hope to find “easy” complete invariants, such as (ordered) groups, rings, modules, etc. An example of such relation is, for example, the relation of isomorphism of AF algebras or Kirchberg algebras.

In [3], John Phillips proved that the relation $E_A^{u.e.}$ of unitary equivalence of automorphisms of a separable non-continuous trace C*-algebra is not smooth. The main result presented here is the following one, which is a strengthening of Phillips' result.

Theorem. *If A is a non-continuous trace separable C*-algebra, then the relation $E_A^{u.e.}$ of unitary equivalence of automorphisms of A is not classifiable by countable structures.*

The main tool in the proof of the theorem is the following non-claissifiability criterion.

Criterion. *Suppose that E is an analytic equivalence relation on the standard Borel space X. Assume moreover that there is a Borel function $f : (0,1)^{\mathbb{N}} \to X$*

such that, for any $x, y \in (0,1)^{\mathbb{N}}$*, if* $x - y \in \ell_1$*, then* $f(x) E f(y)$*, and for any comeager subset* C *of* $(0,1)^{\mathbb{N}}$ *there are* $x, y \in C$ *such that* $f(x) \not{E} f(y)$*. Then,* E *is not classifiable by countable structures.*

The proof of this criterion can be deduced from Hjorth's theory of turbulence and, in particular, from the fact that the action of ℓ_1 on $\mathbb{R}^{\mathbb{N}}$ by translation is turbulent (cf. Proposition 3.25 of [1]). An introduction to the theory of turbulence can by found in [1] (Chapter 3).

In [3], Phillips also shows also that, if A is a unital C*-algebra with continuous trace, then the relation $E_A^{u.e.}$ is smooth. Together with the main result here, this implies that there is a dichotomy in the Borel complexity of the relation of unitary equivalence of automorphisms of a separable unital C*-algebra A: Either such a relation is smooth, or it is non classifiable by countable structures. It would be interesting to know if the same dichotomy holds for non-unital C*-algebras.

Problem 1. *Suppose that* A *is a separable C*-algebra such that* $E_A^{u.e.}$ *is non-smooth. Is it true that* $E_A^{u.e.}$ *is non-classifiable by countable structures?*

It should be observed that, in the non-unital setting, continuous trace does not imply smoothness of the relation of unitary equivalence of automorphisms. There is in fact an example of a separable C*-algebra with continuous trace A such that $E_A^{u.e.}$ is even not classifiable by countable structures. It would be interesting to know exactly for which C*-algebras A the relation $E_A^{u.e.}$ is smooth or, respectively, classifiable by countable structure.

Problem 2. *Characterize the C*-algebras* A *for which* $E_A^{u.e.}$ *is smooth or, respectively, classifiable by countable structures.*

References

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[3] John Phillips, *Outer automorphisms of separable* C^**-algebras*, J. Funct. Anal. **70** (1987), no. 1, 111–116. MR 870756 (88g:46067)