Borel complexity and automorphisms of C*-algebras

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Borel reducibility

E and F are equivalence relations on standard Borel spaces X and Y

$$E \leq_B F \quad \text{iff} \quad \exists f : X \to Y \text{ Borel such that } xEy \text{ iff } f(x)Ff(y)$$

Borel reduction allows one to compare the complexity of eq. relations

This gives a hierarchy in the complexity of classification problems.

Hierarchy

Some equivalence relations can be used as benchmarks of complexity:

- the identity relation id_X on a standard Borel space X;
- the orbit equivalence relation $E^X_{S_\infty}$ induced by some $S_\infty \curvearrowright X$;
- the orbit equivalence relation E^X_G induced by some $G \curvearrowright X$.

An equivalence relations E is:

- smooth if $E \leq_B id_X$ for some X;
- classifiable (by countable structures) if $E \leq E_{S^X_\infty}$ for some $S_\infty \curvearrowright X$;
- orbitable if $E \leq E^X_G$ for some $G \curvearrowright X$.

Conjugacy of actions

Suppose that

- Γ is a countable discrete groups, and
- X is a compact metrizable space.

Define $\mathrm{Homeo}(X)$ to be the group of auto-homeomorphisms of X

An action $\Gamma \curvearrowright_{\phi} X$ is a group homomorphism $\phi : \Gamma \to \mathrm{Homeo}(X)$

Two actions ϕ, ϕ' are conjugate if there is $\alpha \in \mathrm{Homeo}(X)$ such that

$$\alpha \circ \phi(\gamma) \circ \alpha^{-1} = \phi'(\gamma)$$

for every $\gamma \in \Gamma$

Problem: Classify the actions of Γ on X up to conjugacy

$\mathbb{Z}$-actions

Consider the particular case $\Gamma = \mathbb{Z}$

$\mathbb{Z}$-actions on X correspond to single auto-homeomorphisms of X.

Conjugacy of actions correspond to conjugacy in $\mathrm{Homeo}(X)$

Problem: Classify elements of $\mathrm{Homeo}(X)$ up to conjugacy

Polish group of auto-homeomorphisms

$\mathrm{Homeo}(X)$ is a Polish group endowed with the compact-open topology

This is the topology of uniform convergence w.r.t. any compatible metric

We want to study the complexity of the relation

$$E_{\mathrm{Homeo}(X)}$$

of conjugacy inside $\mathrm{Homeo}(X)$

The zero-dimensional case

If X is zero-dimensional then

$$\mathrm{Homeo}(X) \cong \mathrm{Aut}(\mathcal{B}_X)$$

where $\mathcal{B}_X$ is the countable Boolean algebra of clopen subsets of X
Moreover

$$\mathrm{Aut}(\mathcal{B}_X) \subset S_\infty$$

It follows that

$$E_{\mathrm{Homeo}(X)} \leq_B E^X_{S_\infty}$$

for some $S_\infty \curvearrowright X$ and hence $E_{\mathrm{Homeo}(X)}$ is classifiable

The Cantor space

Theorem (Camerlo-Gao, 2001)

If $X = 2^\omega$ is the Cantor space, then

$$E_{\mathrm{Homeo}(2^\omega)}$$

is Borel complete

$E_{\mathrm{Homeo}(2^\omega)}$ has thus maximal complexity among classifiable e.r.

This is obtained of the corollary of

Theorem (Camerlo-Gao, 2001)

Isomorphism of countable Boolean algebras is Borel complete

Equivalently, by (a Borel version of) Stone's duality, homeomorphism of Boolean spaces is Borel complete

Interval and square

Consider now $\mathrm{Homeo}([0,1])$ where $[0,1]$ is the unit interval

Theorem (Hjorth, 2000)

$E_{\mathrm{Homeo}([0,1])}$ *is classifiable*

The proof involves a suitable coding of auto-homeomorphisms of $[0,1]$ by countable structures

On the other hand increase of dimension entails an increase of complexity

Theorem (Hjorth, 2000)

$E_{\mathrm{Homeo}\left([0,1]^2\right)}$ *is not classifiable*

This is one of the earliest application of Hjorth's theory of turbulence

(Pre)turbulent Polish group action

Suppose that

- $G \curvearrowright X$ is a Polish group action
- V is an open neighborhood of 1 in G
- U is an open subset of X
- x is an element of U

The local orbit $\mathcal{O}(x, U, V)$ is the set of points of U that can be reached from x by applying elements of V without ever leaving U

The action $G \curvearrowright X$ is preturbulent if

1. every orbit is dense
2. every local orbit is somewhere dense

The action $G \curvearrowright X$ is turbulent if moreover it has meager orbits

Hjorth's turbulence theorem

Theorem (Hjorth, 2000)

If

- $G \curvearrowright X$ *is preturbulent, and*
- $S_\infty \curvearrowright Y$ *is an* S_∞*-action on a Polish space*

then E_G^X *is generically* $E_{S_\infty}^Y$*-ergodic:*

For any Borel map $f : X \to Y$ *sending* G*-orbits into* S_∞*-orbits,*
$\exists C$ *invariant comeager set such that* $f[C]$ *is contained in a single orbit.*

In particular if $G \curvearrowright X$ is turbulent then E_G^X is not classifiable

Consider

- the Polish group $H = \prod_{\mathbb{N}} \mathbb{Z}$ with the product topology
- the subgroup G of H defined by

$$(x_n) \in G \quad \text{iff} \quad \frac{x_n}{n} \to 0$$

- the action $G \curvearrowright H$ by translation

The action $G \curvearrowright H$ is turbulent.
Moreover any element h of H codes $\phi_h \in \mathrm{Aut}\left([0,1]^2\right)$ such that

$$Gh = Gh' \quad \text{iff} \quad \phi_h \text{ and } \phi_{h'} \text{ are conjugate.}$$

This shows that

$$E_G^H \leq_B E_{\mathrm{Homeo}([0,1]^2)}$$

In particular $E_{\mathrm{Homeo}([0,1]^2)}$ is not classifiable

C*-algebras

A unital C*-algebra A is a complex unital algebra with involution $x \mapsto x^*$ endowed with a norm making it a Banach space and satisfying

$$\|xy\| \leq \|x\| \, \|y\|$$

and

$$\|x^* x\| = \|x\|^2 .$$

All C*-algebras are henceforth assumed to be unital and separable

Suppose that X is a compact metrizable space

Let $C(X)$ be the space of complex-valued continuous functions on X

$C(X)$ is a C*-algebra with pointwise operations and sup norm

Abelian C*-algebras are exactly the ones of this form (Gelfand-Neimark)

Automorphisms groups

Suppose that A is a C*-algebra

Automorphisms of A are bijections $\alpha : A \to A$ preserving all the operations

Automorphisms of A automatically preserve the norm

The group of automorphisms of A is denoted by $\mathrm{Aut}(A)$

$\mathrm{Aut}(A)$ is a Polish group with the topology of pointwise norm convergence

Abelian case

In the abelian case $A = C(X)$ for some compact metrizable space X

There is an exact correspondence between auto-homeomorphisms of X and automorphisms of $C(X)$

Precisely

$$\begin{array}{rcl} \mathrm{Homeo}(X) & \rightarrow & \mathrm{Aut}\,(C(X)) \\ \phi & \mapsto & (f \mapsto f \circ \phi) \end{array}$$

is an isomorphism of Polish groups

In general $\mathrm{Aut}(A)$ can be thought as the group of auto-homeomorphisms of a noncommutative space

Automorphisms of standard probability spaces

Suppose that (Y, ν) is a standard probability space

$\mathrm{Aut}(Y, \nu)$ is the group of measure preserving transformations of Y

The weak topology on $\mathrm{Aut}(Y, \nu)$ is the weakest topology such that

$$T \mapsto \nu\left(T[A] \bigtriangleup B\right)$$

measurable, for every $A, B \subset X$ Borel

$\mathrm{Aut}(Y, \nu)$ is a Polish group endowed with the weak topology

Problem: Study the Borel complexity of conjugacy in $\mathrm{Aut}(Y, \nu)$

The atomless standard probability space

Suppose that (X, μ) is the atomless standard probability space

$$A(X, \mu) = \{T \in \mathrm{Aut}(X, \mu) : T \text{ is aperiodic}\}$$

$T \in \mathrm{Aut}(X, \mu)$ is aperiodic if $T^n \neq 1$ for every $n \geq 1$

Lemma

$A(X, \mu)$ is a conjugation-invariant dense G_δ in $\mathrm{Aut}(X, \mu)$

Theorem (Foreman-Weiss 2004, Kechris 2010)

The action of $\mathrm{Aut}(X, \mu)$ on $A(X, \mu)$ is turbulent

Corollary

The elements of $\mathrm{Aut}(X, \mu)$ are not classifiable up to conjugacy.

Meager conjugacy classes

The shortest argument is due to Rosendal (2009)

Consider for V neighborhood of 1 in $\mathrm{Aut}\,(X,\mu)$ and $I \subset \mathbb{N}$ infinite

$$N(V,I) = \{T \in \mathrm{Aut}\,(X,\mu) : T^n \in V \text{ for some } n \in I\}$$

By Rohlin's lemma $N(V,I)$ is dense open in $\mathrm{Aut}\,(X,\mu)$.
It follows that

$$N(I) = \bigcap_V N(V,I)$$

is a conjugation-invariant dense G_δ subset of $\mathrm{Aut}(X,\mu)$
If C is a nonmeager orbit then $C \subset N(I)$ for every $I \subset \mathbb{N}$ infinite. Thus

$$C \subset \bigcap_{I \subset \mathbb{N} \text{ infinite}} N(I) = \{1\}$$

The Rosendal property

A similar argument works in general.

Suppose that:

- G is a Polish group;
- V is a neighborhood of 1 in G;
- $I \subset \mathbb{N}$ is infinite.

As before set

$$N(I, V) = \{g \in G : g^n \in V \text{ for some } n \in I\}$$

Definition

G has the Rosendal property if $N(I, V)$ is dense for every V and I

Lemma

If

- $G \subset H$ *are Polish groups,*
- G *has the Rosendal property,*

then the relation on G *of conjugacy in* H *does not have a comeager class*

Corollary

Under the assumptions of the theorem, if moreover the action of G *on itself by conjugation is (generically) turbulent, then the elements of* H *are not classifiable up to conjugacy.*

Not purely atomic standard probability space

Suppose that (Y, ν) is a not purely atomic standard probability space.

There is non-null $X \subset Y$ such that

$$(X, \nu_X)$$

is the atomless standard probability space, where

$$\nu(A) = \frac{1}{\nu(X)}\nu(A)$$

for $A \subset X$

Since

$$\mathrm{Aut}(X, \nu_X) \subset \mathrm{Aut}(Y, \nu)$$

it follows that conjugacy in $\mathrm{Aut}(Y, \nu)$ is not classifiable

L^∞ spaces

Define

$$L^\infty(Y,\nu) = \{f : Y \to \mathbb{C} \text{ Borel, bounded, up to mesure zero}\}$$

This is a C*-algebra with respect to pointwise operations, with norm

$$\|f\| = \inf\{\alpha \in \mathbb{R}_+ : f \leq \alpha \text{ a.e.}\}$$

The function

$$\begin{aligned} \tau : L^\infty(Y,\nu) &\to \mathbb{C} \\ f &\mapsto \int f d\nu \end{aligned}$$

defines the norm

$$\|f\|_\tau = \tau(f^* f)^{\frac{1}{2}} = \left(\int |f|^2 d\nu\right)^{\frac{1}{2}}$$

$(L^\infty(Y,\nu),\tau)$ is an example of tracial von Neumann algebra

Traces

A trace τ on a C*-algebra A is a function

$$\tau : A \to \mathbb{C}$$

such that

1. τ is linear
2. $\tau(1) = 1$
3. $\tau(a^* a) \geq 0$
4. $\tau(ab) = \tau(ba)$

τ defines a the τ-norm

$$\|a\|_\tau = \tau(a^* a)^{\frac{1}{2}}$$

Tracial von Neumann algebras

A tracial von Neumann algebra (M, τ) is given by:

1. a C*-algebra M
2. a trace τ on M such that the unit ball of M is complete in τ-norm

The abelian tracial vN algebras are exactly the ones of the form $L^\infty(Y, \nu)$

Tracial von Neumann algebras will be assumed to be separable in τ-norm

Automorphism groups

Suppose that (M, τ) is a tracial von Neumann algebra

Denote by $\mathrm{Aut}(M, \tau)$ the group of automorphisms of M that preserve τ

$\mathrm{Aut}(M, \tau)$ with the topology of pointwise τ-norm convergence is Polish

$\mathrm{Aut}\left(L^{\infty}(Y, \nu), \tau\right) \cong \mathrm{Aut}(Y, \nu)$ (as Polish groups)

In general $\mathrm{Aut}(M, \tau)$ can be thought as the group of
measure preserving transformations of a noncommutative probability space.

Finite factors

Suppose that (M, τ) is a tracial von Neumann algebra

The center of M is

$$Z(M) = \{a \in M : ab = ba \text{ for all } b \in M\}$$

A tracial von Neumann algebra M is a finite factor if the center is trivial

Every tracial vN algebra is isomorphic to a direct integral of factors

The trace in a finite factor is unique

Any automorphism of a finite factor is automatically trace-preserving

II_1 factors

A matrix algebra $\mathbb{M}_n$ with the usual normalized trace is a finite factor

All finite dimensional finite factors are isomorphic to matrix algebras

Definition

A II_1 factor is an infinite dimensional finite factor

II_1 factors play a key role in the theory of von Neumann algebras

They were first studied in the 1930s by Murray and von Neumann

The hyperfinite II_1 factor

Consider the increasing sequence of matrix algebras

$$(\mathbb{M}_{2^n})_{n\in\omega}$$

regarded as finite factors , where

$$\begin{aligned} \mathbb{M}_{2^n} &\subset \mathbb{M}_{2^{n+1}} \\ A &= \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \end{aligned}$$

Define

$$\mathcal{R} = \overline{\bigcup_{n\in\omega} \mathbb{M}_{2^n}} \text{ (completion w.r.t. } \|\cdot\|_\tau \text{)}$$

This is the hyperfinite II_1 factor

It is the Fraïssé limit of the class of matrix algebras (as finite factors)

McDuff II_1 factors

The hyperfinite II_1 factor plays a key role in the study of II_1 factors.

It is the first example of II_1 factor considered by Murray and von Neumann

$\mathcal{R}$ is self-absorbing, i.e. $\mathcal{R} \otimes \mathcal{R} \cong \mathcal{R}$

If M is any II_1 factor, then $\mathcal{R} \subset M$

Definition

A II_1 factor M is McDuff if $M \otimes \mathcal{R} \cong M$

Being McDuff is equivalent to many other regularity properties

Goal: Show that the automorphisms of M are not classifiable

Inspired by the commutative case, let us consider first the case $M = \mathcal{R}$

Inner automorphisms

We need to identify the right analog of aperiodic transformation

Definition

An element u of $\mathcal{R}$ is unitary if $uu^* = u^*u = 1$

Any unitary element defines an automorphism of $\mathcal{R}$

$$\mathrm{Ad}(u) : a \mapsto uau^*$$

Automorphisms of this form are called inner

Remark: In the abelian case there are no nontrivial inner automorphisms

Lemma

Inner automorphisms are dense in $\mathrm{Aut}(\mathcal{R})$

Aperiodic automorphisms

Definition

An automorphism of $\mathcal{R}$ is aperiodic if none of its powers is inner

This is the analogue of aperiodic element of $\mathrm{Aut}\,(X, \mu)$

Denote by $A(\mathcal{R})$ the set of aperiodic elements of $\mathrm{Aut}(\mathcal{R})$

Lemma

$A(\mathcal{R})$ is a conjugation-invariant dense G_δ subset of $\mathrm{Aut}(\mathcal{R})$

Theorem (Kerr-Li-Pichot, 2010)

The action of $\mathrm{Aut}(\mathcal{R})$ on $A\,(\mathcal{R})$ is turbulent

Corollary

Automorphisms of $\mathcal{R}$ are not classifiable up to conjugacy

Nonclassification for McDuff factors

In order to show that the conjugacy classes in $\mathrm{Aut}(\mathcal{R})$ are meager, one in fact shows that $\mathrm{Aut}(\mathcal{R})$ has the Rosendal property

Therefore whenever G is a Polish group such that $G \supset \mathrm{Aut}(\mathcal{R})$, the relation of conjugacy in G is not classifiable

In particular if $M \cong M \otimes \mathcal{R}$ is McDuff then

$$\mathrm{Aut}(M) \cong \mathrm{Aut}\left(M \otimes \mathcal{R}\right) \supset \mathrm{Aut}\left(1 \otimes \mathcal{R}\right) \cong \mathrm{Aut}\left(\mathcal{R}\right)$$

Thus the automorphism of any McDuff II_1 factor are not classifiable

The same conclusion holds for the free group factors $L\left(\mathbb{F}_n\right)$

Open problems

Suppose that M is an arbitrary II_1 factor

Question

Are the automorphisms of M not classifiable up to conjugacy?

Theorem (Kerr-Li-Pichot, 2010)

The action of $\mathrm{Aut}(M)$ on itself by conjugation is generically preturbulent

To answer the question affirmatively, it would be enough to show that $\mathrm{Aut}(M)$ has meager conjugacy classes

C*-algebras (joint work with David Kerr and Chris Phillips)

A is a C*-algebra, and $\mathrm{Aut}(A)$ is the Polish group of automorphisms of A

Problem: Determine the Borel complexity of conjugacy in $\mathrm{Aut}(A)$

The Jiang-Su algebra $\mathcal{Z}$ plays here an analog role as $\mathcal{R}$ for II_1 factors

$\mathcal{Z}$ was defined by Jiang and Su in 1999 in a groundbreaking paper

$\mathcal{Z}$ was constructed as direct limit of suitable "building blocks"

$\mathcal{Z}$ is in fact the Fraïssé limit of these building blocks

$\mathcal{Z}$ is self-absorbing, i.e. $\mathcal{Z} \otimes \mathcal{Z} \cong \mathcal{Z}$ (more in fact is true...)

$\mathcal{Z}$-absorbing C*-algebras

The analogue of McDuff factors are $\mathcal{Z}$-absorbing C*-algebras

A C*-algebra A is $\mathcal{Z}$-absorbing if $A \otimes \mathcal{Z} \cong A$

Conjecture (Toms-Winter)

If A is simple nuclear, the following statements are equivalent

1. *A is $\mathcal{Z}$-absorbing;*
2. *A has finite nuclear dimension;*
3. *A has strict comparison.*

Nuclear dimension is the “right” noncommutative generalization of the notion of dimension for compact metrizable spaces

The Toms-Winter conjecture has been verified in many interesting cases

Strongly aperiodic automorphisms

$\mathcal{Z}$ is endowed with a unique trace τ

The completion $\widehat{\mathcal{Z}}$ of $\mathcal{Z}$ with respect to the τ-norm is isomorphic to $\mathcal{R}$

Any $\alpha \in \mathrm{Aut}(\mathcal{Z})$ induces an automorphism $\widehat{\alpha}$ of $\widehat{\mathcal{Z}} \cong \mathcal{R}$

Definition

α is strongly aperiodic if $\widehat{\alpha} \in \mathrm{Aut}(\mathcal{R})$ is aperiodic

Define $A(\mathcal{Z})$ to be the set of strongly aperiodic automorphisms

Lemma (Sato)

$A(\mathcal{Z})$ is a conjugation-invariant dense G_δ subset of $\mathrm{Aut}(\mathcal{Z})$

Theorem

The action of $\mathrm{Aut}(\mathcal{Z})$ on $A(\mathcal{Z})$ is turbulent

Nonclassification

In order to show that $\mathrm{Aut}(\mathcal{Z})$ has meager conjugacy classes, one in fact shows that $\mathrm{Aut}(\mathcal{Z})$ is a Rosendal group

Therefore whenever G is a Polish group such that $G \supset \mathrm{Aut}(\mathcal{Z})$ the relation of conjugacy in G is not classifiable

In particular if A is a $\mathcal{Z}$-absorbing C*-algebra then

$$\mathrm{Aut}(A) \cong \mathrm{Aut}\,(A \otimes \mathcal{Z}) \supset \mathrm{Aut}\,(1 \otimes \mathcal{Z}) = \mathrm{Aut}(\mathcal{Z})$$

and the automorphisms of A are not classifiable up to conjugacy

The same conclusion holds when $A \cong A \otimes \mathbb{K}$
(consequence of a result of Kechris and Sofronidis about the unitary group)

Open problem

Suppose that A is an infinite-dimensional simple C*-algebra

Question

Is it true that the automorphisms of A are not classifiable up to conjugacy

The assumption that A is simple is necessary

There is a nonsimple A for which automorphisms are classifiable

An example is given by $A = C(2^\omega)$, as we have seen

Outer conjugacy

Consider the relation E of outer conjugacy on $\mathrm{Aut}(\mathcal{Z})$ defined by

$$\alpha E \beta \quad \text{iff} \quad \mathrm{Ad}(u) \circ \gamma \circ \alpha \circ \gamma^{-1} = \beta$$

for some $\gamma \in \mathrm{Aut}(\mathcal{Z})$ and $u \in \mathcal{Z}$ unitary

Question

Is E classifiable by countable structures?

All strongly aperiodic automorphisms of $\mathcal{Z}$ are E-equivalent (Sato, 2009)

Since strongly aperiodic automorphisms form are a comeager set, the theory of turbulence seems not applicable to this situation