Conjugacy and cocycle conjugacy of automorphisms of $\mathcal{O}_2$ are not Borel (joint work with Eusebio Gardella)

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Actions of countable groups on C*-algebras

Let G be a countable group, and A be a separable unital C*-algebra

$\mathrm{Aut}\,(A)$ denotes the group of automorphisms of A

An action $G \curvearrowright_\alpha A$ is a group homomorphism $\alpha : G \to \mathrm{Aut}\,(A)$

Conjugacy and cocycle conjugacy

Suppose that $G \curvearrowright_\alpha A$ and $G \curvearrowright_\beta A$ are actions

Definition

α and β are conjugate if there is $\gamma \in \mathrm{Aut}(A)$ such that

$$\gamma\alpha_g = \beta_g\gamma$$

for $g \in G$.

Definition

Say that α and β are cocycle conjugate if

$$\alpha^u \text{ and } \beta \text{ are conjugate}$$

for some cocycle u.

Problems:

- classify actions $G \curvearrowright_\alpha A$ up to conjugacy
- classify actions $G \curvearrowright_\alpha A$ up to cocycle conjugacy
- classify (simple) crossed products $A \rtimes_\alpha G$

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Obstructions to classification

There might be intrinsic obstructions to perform such classification in a constructive way, and using simple invariants

Definition

A classification is constructive when it is witnessed by a Borel map

This covers all satisfactory classification results in mathematics

Theorem (Farah, Toms, Törnquist, IMRN, 2012)

The computation of the Elliott invariant is given by a Borel map

Borel complexity theory detects and quantifies obstructions to classification

Theorem (Farah, Toms, Törnquist, Crelle's, 2014)

Simple AI algebras are not classifiable using as invariants countable structures –such as countable (dimension) groups, rings, etc–

Nonclassification of automorphisms

Theorem (Kerr, L.,Phillips,2013)

If A is $\mathcal{Z}$-absorbing, then actions $\mathbb{Z} \curvearrowright_\alpha A$ are not classifiable up to conjugacy using countable structures as invariants.

Theorem (L.,2013)

Actions $\mathbb{Z} \curvearrowright_\alpha A$ are not classifiable up to unitary equivalence using countable structures as invariants if (and only if) A is not a finite direct sum of homogeneous C-algebras.*

This in particular rules out classification by K-theoretic data.
This does not rule out a spectral-theorem-type classification.

Theorem (Kechis-Sofronidis, Erg. Th. Dyn. Syst., 2001)

The invariants used in the spectral theorem transcend countable structures

Open problems

Problems:

1. What about classification up to cocycle conjugacy?
2. What about classification of (simple) crossed products?

With E. Gardella we looked at these problems for the Cuntz algebra $\mathcal{O}_2$.

Why is this interesting?

Borel complexity theory offers a framework for the classification program.

It makes "obstructions to classification" precise and quantifiable.

It offer tools to detect and exhibit such obstructions to classifications.

In other words, it tells us what we can and what we can not do.

Classification problems as equivalence relations

Definition

A classification problem is an equivalence relation on a Polish space

$\mathrm{Aut}(A)$ is a Polish group w.r.t. the topology of pointwise convergence

The space $\mathrm{Act}(G, A)$ of actions $G \curvearrowright A$ is a Polish subspace of $\mathrm{Aut}(A)^G$

Example

Conjugacy and cocycle conjugacy are equivalence relations on $\mathrm{Act}(G, A)$
Similarly for the relation of "inducing isomorphic crossed products"

Borel equivalence relations

Definition

An equivalence relation E on a Polish space X is Borel if it is a Borel subset of $X \times X$.

Many natural equivalence relations are Borel.

Example

Isomorphism of finitely generated structures is Borel.

E.g. isomorphism of finitely generated (dimension) groups, rings, etc.

Example

Measure equivalence of Borel measures on $\mathbb{T}$ is Borel.

This is the type of invariants used in the spectral theorem.

Non Borel relations are (somewhat) intractable

Fact

Showing that an equivalence relation is not Borel rules out

1. *classification by finitely generated structures,*
2. *spectral-theorem-type classification, and*
3. *more generally classification by Borel invariants*

Theorem (Camerlo, Gao, TAMS, 2001)

Conjugacy of homeomorphisms of the Cantor space is not Borel.

Theorem (Foreman, Rudolph, Weiss, Annals, 2011)

Conjugacy of ergodic measure preserving transformation of the Lebesgue measure space is not Borel.

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Actions on the Cuntz algebra $\mathcal{O}_2$

Theorem (Gardella, L., 2014)

The relations of conjugacy and cocycle conjugacy of actions $\mathbb{Z}/p\mathbb{Z} \curvearrowright \mathcal{O}_2$ are not Borel.

In fact this statement holds replacing $G = \mathbb{Z}/p\mathbb{Z}$ with any (nontrivial) finitely generated abelian group.

Problem

Are the actions $\mathbb{Z}/p\mathbb{Z} \curvearrowright \mathcal{O}_2$ classifiable up to cocycle conjugacy by countable structures?

Theorem (Gardella, L., 2014)

The relation of isomorphisms of simple purely infinite UCT crossed products $\mathcal{O}_2 \rtimes_\alpha \mathbb{Z}/p\mathbb{Z}$ (with trivial K_1) is not Borel.

Idea of the proof for actions $\mathbb{Z}/p\mathbb{Z} \curvearrowright \mathcal{O}_2$

Theorem (Gardella, L., 2014)

The relations of conjugacy and cocycle conjugacy of (Rokhlin dimension 1) actions $\mathbb{Z}/p\mathbb{Z} \curvearrowright \mathcal{O}_2$ are not Borel.

What does this mean?
Borel sets are those that can be constructed

- starting from open sets,
- taking countable intersections,
- and then countable unions,
- and then countable intersections,
- ...
- and so on by transfinite recursion

Each Borel set is obtained at a given stage after countably (but possibly transfinitely) many steps.

Open sets in the space of (pairs of) actions

A basic open set in the space of pairs (α, β) of actions $\mathbb{Z}/p\mathbb{Z} \curvearrowright \mathcal{O}_2$ is of the form

$$\{(\alpha, \beta) : \|\alpha(x_i) - y_i\| < \varepsilon, \ \|\beta(x_i) - z_i\| < \varepsilon, \ i = 1, 2, \ldots, n\}$$

for $x_i, y_i, z_i \in \mathcal{O}_2$ and $\varepsilon > 0$.

Asserting that (α, β) belongs to such open sets corresponds to specifying the values of α and β on a finite set up to a certain error ε.

A reformulation of the main theorem

Theorem (Reformulation)

There does not exist a procedure or algorithm that

- *takes as input a pair* (α, β) *of actions* $\mathbb{Z}/p\mathbb{Z} \curvearrowright \mathcal{O}_2$
- *runs for countably (possibly transfinitely many) steps*
- *at each step evaluating* α *and* β *on a given (arbitrarily large) finite set*
- *up to an (arbitrarily small) strictly positive error*
- *and at the end of the computation decides whether* α *and* β *are (cocycle) conjugate or not.*

Borel reducibility

The proof relies on the notion of Borel reduction.

Suppose that E is an equivalence relation on X,
and F is an equivalence relation on Y

Definition

E is Borel reducible to F ($E \leq_B F$) if there is a Borel $f : X \to Y$ (called Borel reduction) such that $\forall x, x' \in X$

$$xEx' \text{ iff } f(x)\, F f(x')$$

Fact

If $E \leq_B F$ and E is not Borel, then F is not Borel.

Strategy of the proof

We want to find an equivalence relation E that is not Borel, and a Borel reduction from E to (cocycle) conjugacy of actions $\mathbb{Z}/p\mathbb{Z} \curvearrowright \mathcal{O}_2$

This is done constructing a "Borel inverse" to the map that assigns to an automorphism the K-theory of the crossed product.

Theorem (Downey-Montalbán, after Hjorth)

The relation of isomorphism of torsion free abelian groups is not Borel.

The same argument shows that:

Fact

The relation of isomorphism of countable torsion-free p-divisible self-absorbing (i.e. such that $G \cong G \oplus G$) abelian groups G is not Borel.

Constructing Kirchberg algebras with given K-theory

Lemma

There is a Borel map

$$G \mapsto A_G$$

assigning to a countable discrete group G a simple UCT Kirchberg algebra with K-theory

$$K_*(A_G) \cong (G, 0, \{0\})$$

The construction is an effective version of an argument of Rørdam.

Model actions

Theorem (Izumi, Duke, 2004)

There is an action $\mathbb{Z}/2\mathbb{Z} \curvearrowright_\nu \mathcal{O}_2$ (of Rokhlin dimension 1) such that

$$K_*\left(\mathcal{O}_2 \rtimes_\nu \mathbb{Z}/\mathbb{Z}_2\right) \cong \left(\mathbb{Z}\left[\frac{1}{2}\right], 0, \{0\}\right)$$

Theorem (Barlak, Szabó, 2014)

For $p > 2$ there is an action $\mathbb{Z}/p\mathbb{Z} \curvearrowright_\nu \mathcal{O}_2$ (of Rokhlin dimension 1) such that

$$K_*\left(\mathcal{O}_2 \rtimes_\nu \mathbb{Z}/p\mathbb{Z}\right) \cong \left(\mathbb{Z}\left[\frac{1}{p}\right] \oplus \mathbb{Z}\left[\frac{1}{p}\right], 0, \{0\}\right)$$

Using the model action one can define the action

$$\nu \otimes id_{A_G} : \mathbb{Z}/p\mathbb{Z} \curvearrowright \mathcal{O}_2 \otimes A_G$$

Fact

The algebra

$$A \rtimes_{\nu \otimes id_{A_G}} (\mathcal{O}_2 \otimes A_G)$$

is a UCT Kirchberg with K-theory

$$\left(\mathbb{Z}\left[\frac{1}{2}\right] \otimes G, 0, \{0\}\right)$$

for $p = 2$ *and*

$$\left(\left(\mathbb{Z}\left[\frac{1}{p}\right] \oplus \mathbb{Z}\left[\frac{1}{p}\right]\right) \otimes G, 0, \{0\}\right)$$

for $p > 2$.

From groups to actions

Let G, H be countable groups.

The following statements are in decreasing order of strength

1. $A_G \cong A_H$
2. $\nu \otimes id_{A_G}$ and $\nu \otimes id_{A_H}$ are conjugate
3. $\nu \otimes id_{A_G}$ and $\nu \otimes id_{A_H}$ are cocycle conjugate
4. $(\mathcal{O}_2 \otimes A_G) \rtimes_{\nu \otimes id_{A_G}} \mathbb{Z}/p\mathbb{Z} \cong (\mathcal{O}_2 \otimes A_H) \rtimes_{\nu \otimes id_{A_H}} \mathbb{Z}/p\mathbb{Z}$

If G and H are uniquely p-divisible and self-absorbing, then the statements 1–4 are all equivalent.

From group to actions

Theorem

There is a Borel function

$$G \mapsto \alpha_G : \mathbb{Z}/p\mathbb{Z} \curvearrowright \mathcal{O}_2$$

where α_G *is conjugate to* $\nu \otimes id_{A_G} : \mathbb{Z}/p\mathbb{Z} \curvearrowright \mathcal{O}_2 \otimes A_G$.

Such function is a Borel reduction from isomorphism of countable torsion-free p-divisible self-absorbing abelian groups to conjugacy and cocycle conjugacy of automorphisms of $\mathcal{O}_2$

From groups to crossed products

Theorem

The crossed product $\mathcal{O}_2 \rtimes_{\alpha_G} \mathbb{Z}/p\mathbb{Z}$ can be computed in a Borel way.

The function

$$G \mapsto \mathcal{O}_2 \rtimes_{\alpha_G} \mathbb{Z}/p\mathbb{Z}$$

is a Borel reduction from isomorphism of countable torsion-free p-divisible self-absorbing abelian groups G to isomorphism of simple purely infinite UCT crossed products $\mathcal{O}_2 \rtimes_{\alpha} \mathbb{Z}/p\mathbb{Z}$