Groupoid L^p operator algebras (joint work with Eusebio Gardella)

Martino Lupini

York University and Fields Institute

June 26, 2014

Table of Contents

Table of Contents

L^p spaces

λ is a Borel σ-finite measure on a standard Borel space Z

We will assume for convenience that λ is not purely atomic

$L^p(\lambda)$ is the Banach space of p-summable functions on Z up to null sets

$B\left(L^p(\lambda)\right)$ is the Banach algebras of bounded linear operators on $L^p(\lambda)$

$B\left(L^p(\lambda)\right)$ is matricially normed by

$$M_n\left(B\left(L^p(\lambda)\right)\right) \cong B\left(L^p\left(\lambda \times c_n\right)\right)$$

where c_n is the counting measure on $\{0, 1, \ldots, n-1\}$

L^p operator algebras

Definition

A concrete L^p operator algebra A is a closed subalgebra of $B(L^p(\lambda))$

A is matricially normed by

$$M_n(A) \subset M_n\left(B\left(L^p\left(\lambda\right)\right)\right) \cong B\left(L^p\left(\lambda \times c_n\right)\right)$$

A is this p-operator system s. t. every $M_n(A)$ is a Banach algebra

Definition

An abstract L^p operator algebra A is a matricially normed Banach algebra completely isometrically isomorphic to a concrete L^p operator algebra

Problem

Is there an abstract (intrinsic) characterization of L^p operator algebras?

Some previous works

The general theory of L^p operator algebras has been recently "launched" by Chris Phillips.

Among the examples he considered there are:

- L^p analogs of the Cuntz C*-algebras
- L^p analogs of the UHF C*-algebras
- enveloping L^p operator algebras of locally compact groups

Subsequent work by Phillips-Viola, Gardella-Thiel, Pooya-Hejazian ...

Problem: generalizing results from C*-algebras to L^p operator algebras

The main difference is that in L^p there is no adjoint for $p \neq 2$

Table of Contents

A groupoid approach

We studied representations of étale groupoids on L^p spaces constructing the associated enveloping L^p operator algebras

Goals:

1. isolate the "good" representations of algebraic objects on L^p
2. give a common generalization of the L^p UHF and Cuntz algebras
3. provide several new example of L^p analogs of "classical" C*-algebras (Cuntz-Krieger algebras, tiling algebras, graph algebras...)

Étale groupoids

A groupoid G is a small category where every arrow is invertible.

The set of objects is denoted by G^0 and identified with a subset of G

Source and range maps are denoted by

$$s, r : G \to G^0$$

A slice of G is a subset A of G such that s and r are 1:1 on A

A groupoid is locally compact when endowed with a locally compact topology making composition and inversion of arrows continuous

A locally compact groupoid is étale if it has a countable basis of open slices

Transformation groupoids

Suppose that

- Γ is a countable group
- X is a locally compact space
- $\Gamma \curvearrowright X$ is an action of Γ on X

The transformation groupoid $X \rtimes \Gamma$ is the set of triples

$$(\gamma x, \gamma, x)$$

for $\gamma \in \Gamma$ and $x \in X$ with composition

$$(\rho\gamma x, \rho, \gamma x)\,(\gamma x, \gamma, x) = (\rho\gamma x, \rho\gamma, x)$$

More generally one can consider transformation groupoids associated with

- local homeomorphism (Cuntz-Krieger and graph groupoids)
- (partial) action of inverse semigroups (these cover all étale groupoids)

The algebra of continuous compactly supported functions

The space $C_c(G)$ of continuous functions is a *normed *-algebra with*

- multiplication by convolution

$$(f_0 * f_1)(\gamma) = \sum_{\rho_0\rho_1=\gamma} f(\rho_0)f(\rho_1)$$

- involution

$$(f^*)(\gamma) = \overline{f(\gamma^{-1})}$$

- norm

$$\|f\|_I = \max\left\{ \sup_{x\in G^0} \sum_{r(\gamma)=x} |f(\gamma)|, \sup_{x\in G^0} \sum_{s(\gamma)=x} |f(\gamma)| \right\}$$

It is in fact matricially normed via the identification

$$M_n(C_c(G)) \cong C_c(n \times G \times n)$$

where $n \times G \times n$ is a suitable amplification of G

Representations of groupoids on bundles of Hilbert spaces

Suppose that G is an étale groupoid.

A representation of G on a Hilbert bundle is given by

1. a quasi-invariant probability Borel measure μ on G^0
2. a Borel collection $(H_x)_{x \in G^0}$ of Hilbert spaces
3. a Borel assignment $\gamma \to T_\gamma$ such that

- T_γ is an invertible isometry from $H_{s(\gamma)}$ to $H_{r(\gamma)}$
- $T_\gamma T_\rho = T_{\gamma\rho}$ a.e.
- $T_{\gamma^{-1}} = T_\gamma^{-1}$ a.e.

Representations of groupoids on bundles of L^2 spaces

In fact without loss of generality there are

1. a standard Borel space Z
2. a Borel surjection $q : Z \to G^0$
3. a σ-finite Borel measure λ on Z with disintegration $\int \lambda_x d\mu(x)$

such that $H_x = L^2(\lambda_x)$ for every $x \in G^0$

Define Z_x to be the inverse image of x under q

Observe that λ_x is a σ-finite Borel measure on $Z_x = p^{-1}\{x\}$ for $x \in G^0$

Moreover if $\xi \in L^2(\lambda)$ then $\xi_{|Z_x} \in L^2(\lambda_x)$ for μ-a.e. $x \in G^0$

Integrated form of a representation

Consider as before the representation

$$\gamma \mapsto T_\gamma : L^2\left(\lambda_{s(\gamma)}\right) \to L^2\left(\lambda_{r(\gamma)}\right)$$

Is integrated form is the I-norm contractive *-homomorphism

$$\pi : C_c(G) \to B\left(L^2(\lambda)\right)$$

defined by

$$\left(\pi(f)\xi\right)_{|Z_y} = \sum_{r(\gamma)=y} f(\gamma) D^{-\frac{1}{2}}(\gamma)\, T_\gamma \xi_{|Z_{s(\gamma)}}$$

where D is the modular function of (G, μ)

Renault's disintegration theorem

Theorem (Renault, 1980)

Every I-norm contractive nondegenerate representation $\pi : C_c(G) \to B\left(L^2(\lambda)\right)$ *is of this form*

Corollary

Every I-norm contractive nondegenerate homomorphism $\pi : C_c(G) \to B\left(L^2(\lambda)\right)$ *is a *-homomorphism.*

The groupoid C*-algebra $C^*(G)$ is the enveloping C*-algebra of $C_c(G)$

Theorem (Renault, 1980)

There is a correspondence between

1. *representations of* G *on Hilbert bundles*
2. *contractive nondegenerate Hilbert (*-)representations of* $C_c(G)$
3. *contractive nondegenerate Hilbert (*-)representations of* $C^*(G)$

Representations of groupoids on L^p bundles

What happens for representations on L^p spaces?

The notion of representation of G is defined as before, replacing L^2 with L^p

The construction of the integrated form of a representation goes through

Theorem (Gardella, L., 2014)

Every I-norm contractive nondegenerate homomorphism $\pi : C_c(G) \to L^p(\lambda)$ comes from a representation of G

Corollary

Every I-norm contractive nondegenerate homomorphism $\pi : C_c(G) \to B\left(L^p(\lambda)\right)$ is completely contractive

The groupoid L^p operator algebra

The groupoid L^p operator algebra $F^p(G)$ is enveloping algebra of $C_c(G)$ with respect to representations on L^p spaces

Theorem (Gardella, L., 2014)

There is a correspondence between

1. *representations of G on bundles of L^p spaces*
2. *I-norm (completely) contractive nondegenerate representations of $C_c(G)$ on L^p spaces*
3. *I-norm (completely) contractive nondegenerate representations of $F^p(G)$ on L^p spaces*

Table of Contents

The inverse semigroup of clopen slices

Let us consider the case when G^0 is compact zero-dimensional.

The representation theory of G is determined by a purely algebraic object

The collection Σ_G of clopen slices is a basis for G

If $A, B \in \Sigma_G$ define

$$\begin{aligned} AB &= \{\gamma\rho : \gamma \in A, \rho \in B\} \\ A^* &= \{\gamma^{-1} : \gamma \in A\} \end{aligned}$$

This makes Σ_G a countable semigroup such that:
for every $A \in \Sigma_G$ there is a unique $A^* \in \Sigma_G$ such that

$$AA^*A = A \text{ and } A^*AA^* = A^*$$

This means that Σ_G is an inverse semigroup

The idempotent semilattice

Consider the set $E(\Sigma_G)$ of idempotent elements of Σ_G

The elements of $E(\Sigma_G)$ are precisely the clopen subsets of G^0

Thus $E(\Sigma_G)$ is just the Stone Boolean algebra of G^0

Representation of the semigroup of slices

Identify A with its characteristic function $\chi_A \in C_c(G)$

This makes Σ_G a multiplicative subsemigroup of $C_c(G)$

A representation

$$\pi : C_c(G) \to B(H)$$

induces by restriction a semigroup homomorphism from Σ_G to an inverse semigroup of partial isometries of H

Theorem (Renault 1980, Exel 2008)

This establishes a correspondence between

1. *I-norm contractive nondegenerate (*-)representations of $C_c(G)$*
2. *tight homomorphisms from Σ_G to an inverse semigroup of partial isometries of H*

Tightness

Tightness is a nondegeneracy condition introduced by Exel

Definition (Exel, 2008)

A homomorphism from Σ_G to an inverse semigroup of partial isometries of H is tight when it restricts to a Boolean algebra homomorphism from $E(\Sigma_G)$ to a Boolean algebra of projections of H.

This is formulated differently for more general groupoids.

From inner-product spaces to semi-inner product spaces

How to adapt this correspondence to the L^p case?

The notions of

- positive element
- orthogonal projection
- partial isometry

can be defined using the inner product of H

L^p spaces have something similar, called semi-inner product

Semi-inner product spaces

Definition (Lumer, 1961)

A semi-inner product on X is a function

$$[\cdot,\cdot] : X \times X \to \mathbb{C}$$

such that

1. $[\cdot,\cdot]$ is linear in the first variable
2. $[x, \alpha y] = \overline{\alpha}\, [x, y]$
3. $[x, x] \geq 0$, and equality holds iff $x = 0$
4. $|[x, y]| \leq [x, x]^{\frac{1}{2}}\, [y, y]^{\frac{1}{2}}$

This is an inner product precisely when it is linear in the second variable. The associated norm is

$$\|x\| = [x, x]^{\frac{1}{2}}$$

The canonical semi-inner product on L^p spaces

Consider on $L^p(\lambda)$ the semi-inner product

$$[f, g] = \|g\|_p^{2-p} \int \left(f \cdot \overline{g} \cdot |g|^{p-2} \right) d\lambda$$

It is the unique semi-inner product on $L^p(\lambda)$ inducing the usual norm

(All smooth Banach spaces have a unique semi-inner product structure)

Hermitian operators

Suppose that X is a semi-inner product space and $a \in B(X)$

The operator range of a is

$$W(A) = \{[ax, x] : [x, x] = 1\}$$

Theorem (Lumer, 1961)

The following are equivalent:

1. $W(A) \subset \mathbb{R}$
2. $\|1 + ita\| = 1 + o(t)$ *for* $t \to 0$
3. $\|\exp(ita)\| = 1$ *for every* $t \in \mathbb{R}$

In such case a is called hermitian

Example

When $X = H$ and $a \in B(H)$ then a is hermitian iff it is self-adjoint.

Partial isometries L^p

It is natural to replace orthogonal projections with hermitian idempotent operators in $B(L^p(\lambda))$

This leads to the following definition:

Definitions

An operator $a \in B(L^p(\lambda))$ for $p \neq 2$ is a (spatial) partial isometry if there is $b \in B(L^p(\lambda))$ such that

$$ab \text{ and } ba \text{ are hermitian idempotents}$$

The Banach-Lamperti theorem

Partial isometries of L^p spaces have been characterized by Banach (1932). The first available proof is due to Lamperti (1958).

Theorem (Banach-Lamperti)

Partial isometries on $L^p(\lambda)$ for $p \neq 2$ are all of the form

$$f \mapsto g \cdot \left(f_{|A} \circ \phi^{-1}\right)$$

where

1. $A, B \subset Z$ *are Borel*
2. $\phi : A \to B$ *is a measure-class preserving Borel isomorphism*
3. $g : Z \to \mathbb{C}$ *is Borel supported by* B

Corollary ($p \neq 2$)

Partial isometries of $L^p(\lambda)$ form an inverse semigroup $\mathcal{S}\left(L^p(\lambda)\right)$

Back to groupoids

Suppose that G^0 is compact zero-dimensional,
Σ_G is the countable inverse semigroup of clopen slices
Consider for $p \neq 2$ a representation of G

$$\gamma \mapsto T_\gamma : L^p\left(\lambda_{s(\gamma)}\right) \to L^p\left(\lambda_{r(\gamma)}\right)$$

on a bundle of L^p spaces coming from the disintegration $\lambda = \int \lambda_x d\mu(x)$

Consider the homomorphism ρ from Σ_G to $\mathcal{S}\left(L^p(\lambda)\right)$ defined by

$$(\rho(A)\xi)_{|y} = T_\gamma \xi_{|s(\gamma)}$$

where γ is the unique element of A such that $r(\gamma) = y$

This is a tight homomorphism from Σ_G to $\mathcal{S}\left(L^p(\lambda)\right)$

Theorem (Gardella, L., 2014)

Every tight homomorphism $\Sigma_G \to \mathcal{S}\left(L^p\left(\lambda\right)\right)$ is of this form

Theorem (Gardella, L., 2014)

There is a correspondence between

1. *representations of G on bundles of L^p spaces*
2. *I-norm (completely) contractive representations of $C_c(G)$ on $L^p(\lambda)$*
3. *tight homomorphisms from Σ_G to $\mathcal{S}\left(L^p(\lambda)\right)$*

Denote by $F^p_{tight}\left(\Sigma_G\right)$ the enveloping algebra of $\mathbb{C}\Sigma_G$ corresponding to tight homomorphism in $\mathcal{S}\left(L^p(\lambda)\right)$

Corollary

$F^p(G)$ is (completely) isometrically isomorphic to $F^p_{tight}\left(\Sigma_G\right)$

One can define $F^p_{tight}\left(\Sigma\right)$ for any (abstract) inverse semigroup Σ

L^p analogs of Cuntz C*-algebras

Consider the inverse semigroup Σ generated by

$$\sigma_1, \ldots, \sigma_d, \sigma_1^*, \ldots, \sigma_d^*$$

together with a zero 0 and an identity 1 subject to the relations

$$\sigma_i^* \sigma_i = 1$$

$$\sigma_j^* \sigma_i = 0 \text{ for } i \neq j$$

In such case $F^2_{tight}(\Sigma)$ is the Cuntz algebra $\mathcal{O}_d$

while $F^p_{tight}(\Sigma)$ is the Phillips' L^p analog of the Cuntz algebra $\mathcal{O}_d^p$

The Cuntz-Krieger semigroup

Suppose that A is a $d \times d$ matrix with entries in $\{0,1\}$ satisfying Cuntz-Krieger condition (I)

Consider the inverse semigroup Σ_A generated by

$$\sigma_1, \ldots, \sigma_d, \sigma_1^*, \ldots, \sigma_d^*$$

together with a zero 0 subject to the relations

$$\begin{aligned} (\sigma_i^* \sigma_i)(\sigma_j \sigma_j^*) &= A(i,j)(\sigma_j \sigma_j^*) = (\sigma_j \sigma_j^*)(\sigma_i^* \sigma_i) \\ \sigma_j^* \sigma_i &= 0 \text{ for } i \neq j \\ (\sigma_j^* \sigma_j)(\sigma_i^* \sigma_i) &= (\sigma_i^* \sigma_i)(\sigma_j^* \sigma_j) \end{aligned}$$

$F^2_{tight}(\Sigma_A) \cong \mathcal{O}_A$ is a Cuntz-Krieger algebra

$F^p_{tight}(\Sigma_A) = \mathcal{O}^p_A$ can be seen as L^p analog of Cuntz-Krieger algebras

Future work

1. Uniqueness theorems for representations
2. Simplicity
3. Interpolation
4. Quotients
   It is not known if L^p operator algebras are closed by taking quotients
   What about quotients of $F^p(G)$?
5. Second dual
   Any L^p operator algebra is Arens regular (Daws, 2004)
   Moreover A^{**} is again an L^p operator algebra with
   - the Arens product
   - the bidual p-operator space structure